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ABSTRACT

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in summer, 1965. It is intended for college students who are not preparing to become professional physicists. The monograph contains three chapters. Chapter 1 deals with the law of inertia for objects at rest and in motion, the theory of Galilean relativity and deviations from the law of inertia. The law of momentum conservation and its applications are discussed in chapter 2. In chapter 3, the principle of energy conservation and the concept of kinetic energy are discussed. A number of experiments and problems are included in each chapter. (LC)

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Matter in Motion

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GENERAL PREFACE

This monograph was written for the Conference on the New Instructional Materials in Physics, held at the University of Washington in the summer of 1965. The general purpose of the conference was to create effective ways of presenting physics to college students who are not preparing to become professional physicists. Such an audience might include prospective secondary school physics teachers, prospective practitioners of other sciences, and those who wish to learn physics as one component of a liberal education.

At the Conference some 40 physicists and 12 filmmakers and designers worked for periods ranging from four to nine weeks. The central task, certainly the one in which most physicists participated, was the writing of monographs.

Although there was no consensus on a single approach, many writers felt that their presentations ought to put more than the customary emphasis on physical insight and synthesis. Moreover, the treatment was to be "multi-level" --- that is, each monograph would consist of several sections arranged in increasing order of sophistication. Such papers, it was hoped, could be readily introduced into existing courses or provide the basis for new kinds of courses.

Monographs were written in four content areas: Forces and Fields, Quantum Mechanics, Thermal and Statistical Physics, and the Structure and Properties of Matter. Topic selections and general outlines were only loosely coordinated within each area in order to leave authors free to invent new approaches. In point of fact, however, a number of monographs do relate to others in complementary ways, a result of their authors' close, informal interaction.

Because of stringent time limitations, few of the monographs have been completed, and none has been extensively rewritten. Indeed, most writers feel that they are barely more than clean first drafts. Yet, because of the highly experimental nature of the undertaking, it is essential that these manuscripts be made available for careful review by other physicists and for trial use with students. Much effort, therefore, has gone into publishing them in a readable format intended to facilitate serious consideration.

So many people have contributed to the project that complete acknowledgement is not possible. The National Science Foundation supported the Conference. The staff of the Commission on College Physics, led by E. Leonard Jossem, and that of the University of Washington physics department, led by Ronald Geballe and Ernest M. Henley, carried the heavy burden of organization. Walter C. Michels, Lyman G. Parratt, and George M. Volkoff read and criticized manuscripts at a critical stage in the writing. Judith Bregman, Edward Gerjuoy, Ernest M. Henley, and Lawrence Wilets read manuscripts editorially. Martha Ellis and Margery Lang did the technical editing; Ann Widditsch supervised the initial typing and assembled the final drafts. James Grunbaum designed the format and, assisted in Seattle by Roselyn Pape, directed the art preparation. Richard A. Mould has helped in all phases of readying manuscripts for the printer. Finally, and crucially, Jay F. Wilson, of the D. Van Nostrand Company, served as Managing Editor. For the hard work and steadfast support of all these persons and many others, I am deeply grateful.

Edward D. Lambe
Chairman, Panel on the
New Instructional Materials
Commission on College Physics

MATTER IN MOTION

PREFACE

Physics describes the real world. The laws of physics are useful because they describe the behavior of actual objects in real situations. We use the language of mathematics to express the laws, and the logic of mathematics to derive predictions from them. But physics is not axiomatic. The assumptions on which physical laws are based are extracted from the physical world, and the predictions made by the laws must be tested in laboratory experiments.

This monograph draws upon selected experiences from life and from the laboratory to reveal clearly some of the patterns of nature.

The conservation laws for linear momentum and for energy developed herein are relevent to all natural phenomena, and provide insight into a wonderful variety of events. They are not obvious from a superficial examination, however, but require careful training of the observing eye and analyzing mind to discern them. We hope that in learning to recognize this order in nature the reader will also discover some of the beauty we see in her patterns.

From initial observations and experiments we try to formulate general laws which describe many events. Attempts to state the laws precisely in verbal or mathematical form often lead to definite questions which must be answered in the laboratory before the laws can be stated. Once a law is clearly stated, it should make definite predictions about new situations. These predictions can themselves be tested in controlled laboratory experiments. Our confidence in the general applicability of any law depends ultimately on its past success in predicting the results of experiments. The more times the predictions are confirmed, the more confidently we apply the law to tested.

The number and variety of experiments which confirm the predictions of the laws of momentum and energy conservation is so large that what we can present is limited by taste and time rather than by any scarcity of data.

Within each section of the monograph we have attempted to state clearly the principle to be developed. Then we have turned to experiment and description to clarify and sharpen the content of the principle.

The emphasis has been placed here upon the general conservation laws and their identical forms in all inertial frames. The principle of Galilean relativity has been chosen as the expression of this invariance because it seems to us to be the "natural" expression of relativity for elementary mechanics.

The monograph is incomplete in that it lacks the section on Forces. This section, though necessary to a consistent development, would not have been large (as indicated in the block diagram of the monograph below). We hope that with a thorough understanding of these two conservation laws, together with an introduction to the study of interactions through forces and potential energies, the student can study further the interactions of matter in context. We hope he will use the tools of the conservation laws to study the structure of matter, the collisions of elementary particles, the nature of fields, etc., in other parts of his course.

We have assumed that the student is familiar with the vector description of motions before he reads this monograph. A suitable background might be found in the first six chapters of the PSSC text Physics, or the monograph Motion written by Gerhart and Nussbaum at the same time as this one.

The original vision of the monograph on momentum and energy conservation looked something like the sketch below, in which the size of each area is roughly proportional to the number of pages.

Proposed
Law of Inertia
Momentum Conservation
Forces
Energy Conservation

As it turned out, the monograph is not completed. The completed sections are indicated below to roughly the same scale as above.

Done	Missing
LAW OF INERTIA	
Galilean relativity	
MOMENTUM CONSERVATION	
	← Forces
ENERGY CONSERVATION (Kinetic)	← Potential energies' thermal energies, etc.

The "Law of Inertia" and "Conservation of Momentum" chapters are sufficiently complete that they can probably be used in the classroom. Although some "thinking" questions are presented in the text, additional drill problems will be needed for classroom use.

Even a casual reading of the text will make it clear that it is intended to be used along with classroom demonstrations as well as laboratory experiments.

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1 THE LAW OF INERTIA

1.1 INTRODUCTION

The law of inertia describes the tendency of matter in motion to continue. At the mouth of a swift river one can see the clear lake waters parted by the moving, muddy, river water, which continues the motion it had overland until the drag of the neighboring liquid brings it to a stop. An automobile at rest requires a force to put it in motion, another force to turn it. An arrow once put in motion by a bowstring continues until it hits something, or until the effects of air resistance and gravity slow it and pull it to earth.

The law of inertia was developed as an expression of what matter would do "if it were left alone." In terms of the last example, the law of inertia expresses what observations and experimental tests indicate the arrow would do if there were no gravity and no air resistance. The word "until" in the description of the arrow's motion already implies that the motion would in some way continue if there were no outside interference. But we shall try to express this notion in a more precisely stated form that can be tested in the laboratory.

1.2 THE LAW OF INERTIA FOR OBJECTS AT REST

How does an object at rest behave if it is "left alone"? What would it do if it were placed at rest, far from the remainder of the universe? We expect that the object would remain at rest. This expectation is a generalization of the everyday experience that to put an object in horizontal motion at the earth's surface requires some sort of effort, some sort of interaction with another object. Matter is a sluggish sort of thing, hard to get going.

To each object a number called its mass can be assigned to specify this sluggishness. A way to determine quantitatively the mass of an object will be developed in a later chapter. For now, however, it is already convenient to introduce the word. It is common in physics to hear or read the words "a mass" used instead of "an object" when special attention is directed to this sluggish behavior of a bit of matter.

In history the difficulty in discerning the behavior of a mass, i.e., of an object, arose in separating the properties of the mass itself from effects peculiar to the earth's surface. It is still difficult to conceive the idea of a piece of matter, an earthly object, far away from and independent of the surface of the earth. But it is not nearly so difficult now as it was before children heard their parents and elders discuss such things. On the surface of the earth, motions in the vertical and horizontal directions differ; there is gravity. We must decide whether a released object would still fall vertically, i.e., toward the point now at the earth's center, if the earth were not there. To one steeped in the modern view that space itself is homogeneous and isotropic, the natural answer is that there would be no vertical direction in which to fall if the earth were not present.

If we probe the question further, you will see how an attitude toward space influences one's answer to the question. Suppose the object had been placed at rest at point A, and all on its own, with no outside forces, it had moved to point B at the end of one second. It would no longer be possible to assert that all points in space are equivalent (i.e., space would not be homogeneous as far as this object is concerned), unless the object moved always the same amount in the same direction in the first second no matter where it was placed at rest. But then that constant direction would be "special"; i.e., we could not say that space is isotropic.

To one who builds his ideas from common earthly sense experiences, the universe may not seem to be such a featureless thing with no "preferred" point or direction. In building a system to explain the universe, Aristotle assumed that different kinds of matter have natural places to which they tend to return. In his system the earthly matter of familiar solids and liquids tends to move toward the center of the universe. Thus, in his system, space itself has a special point (the center of the universe) which differs from others. This tendency explained to Aristotle why objects fall vertically toward the earth's center, for the earth had presumably already arrived at the center of the universe. Indeed, its compact spherical shape shape was explained as the result of its parts clustering as close to that center as possible.

In the modern view we take the vertical motion of falling to be something peculiar to the presence of the earth in that direction, and take the horizontal behavior of massive objects to be typical of their behavior in all three dimensions if the earth were not present, or if the earth and all other disturbing objects were very far away.

QUESTION

Are these two views of space mutually consistent? In terms of what you now know, can you devise an experiment to decide between them? An ideal experiment realizable in thought rather than in the college laboratory is acceptable here, but eventually the question must be presented in the laboratory.

Although the notion or statement that an isolated object at rest remains at rest is an appealing one if we believe that space is homogeneous and isotropic, the validity of the statement does not depend upon its subjective appeal. The ultimate appeal must be to experiment, even though a direct experimental test of this simple statement is very difficult. We shall, however, be able to put this assumption together with others to build a coherent description of gradually more complicated physical systems. Our trust in this and other basic notions will increase as long as the descriptions enable us to make new verifiable predictions about these systems, and to the extent that the predictions are indeed borne out by experiments.

1.3 THE LAW OF INERTIA FOR OBJECTS IN MOTION

Everyone has experienced the difficulty of stopping or turning, as well as starting, a very massive object in motion. In order to make a ferryboat stop just at the edge of the pier, it is not sufficient to turn off the engines at the instant of arrival. A boulder tumbling down the side of a hill does not stop right at the bottom except in very unusual circumstances; if the ground below is level, it rolls on for some distance. And yet the behavior of an object in motion, isolated from all disturbing effects, is quite difficult to extract from experience.

It was not until the seventeenth century that the simple behavior of matter in motion was clearly described in the Law of Inertia:

> An isolated mass moving with a given velocity will continue with the same velocity as long as it remains isolated.

A constant velocity means both constant speed and constant direction, of course, because velocity is a vector quantity. "An isolated mass" refers to an object alone in space, far away from and therefore free from the effects of any disturbing influences. However, when we test the law of inertia directly in the laboratory, we must be content with systems that are approximately isolated, systems in which we have reduced the disturbing influences of gravity, friction, air resistance, etc., as much as possible.

The motions of objects in everyday experience are so different from the motion of an approximately isolated object that even today the law of inertia does not seem natural to students of physics when they first meet it. Objects moving on the surface of the earth come eventually to a stop if they are "left alone." The reason for this is that the world around them does not really leave them unaffected. The motion of a child's wagon along a sidewalk, for example, is resisted by the bumps in the walk, the air that must be pushed aside, the crunchy dirt, and the slightly sticky oil in the wheel bearings. Snow sledding and ice skating are special fun partly because in these sports the motion persists for a long time without any effort, but even the slight drag of the ice on the runners eventually brings the ride to a halt.

The success of sled runners on ice suggests a way to make horizontal motion at the earth's surface approximate the motion of an object that is really isolated. The pressure of the runners melts some of the ice and provides a thin film of water as a lubricant. Introducing a fluid between two solid surfaces (for example, oil in bearings), is one way of reducing the frictional drag between them. Gases are less sticky than liquids and work even better as lubricants. Many of the most successful demonstrations of the behavior of isolated mechanical systems use a gas film for reducing friction. Figure 1.1 shows some pictures of an air disk, which moves across a smooth flat surface with almost no friction. The motion of the disk is quite well isolated. It is free from friction against the surface on which it rides, and keeping the surface horizontal makes the motion in two dimensions free of the influence of gravity.

When the air disk of Fig. 1.1 is released at rest, it remains at rest on the flat horizontal table, in accord with the predictions of the law of inertia for isolated objects at rest. It may worry you a little if the table were leveled by adjusting it for no motion of the air disk. But is it not wonderful that it is possible to find one level position such that all

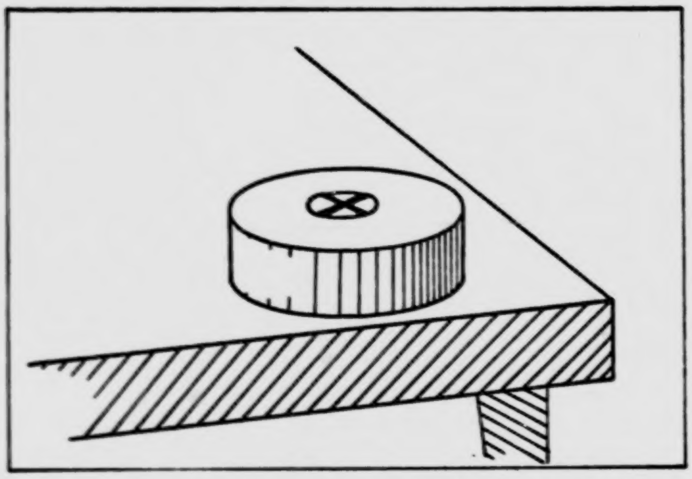

Fig. 1.1a One kind of air disk consists of a Lucite cylinder marked at its center with a cross on a paper circle. The disk rests on an air table. With this apparatus we can study almost frictionless motion.

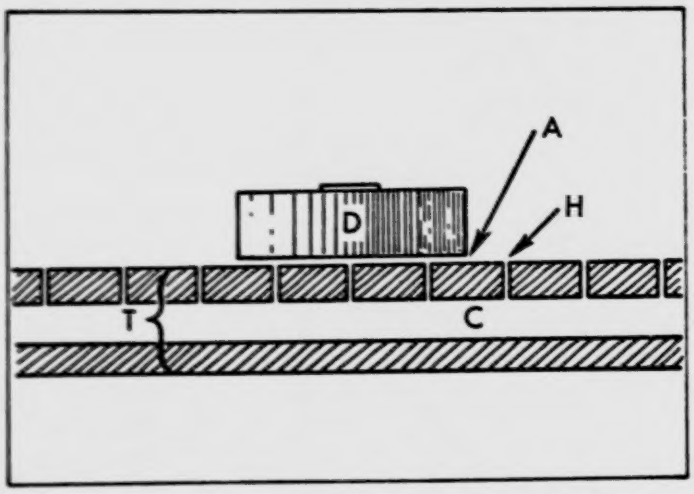

Fig. 1.1b A thin layer of air A flowing from the holes H in the air table T keeps the Lucite disk D floating above the solid surface. The air is fed to the holes through channels C cut into the air table.

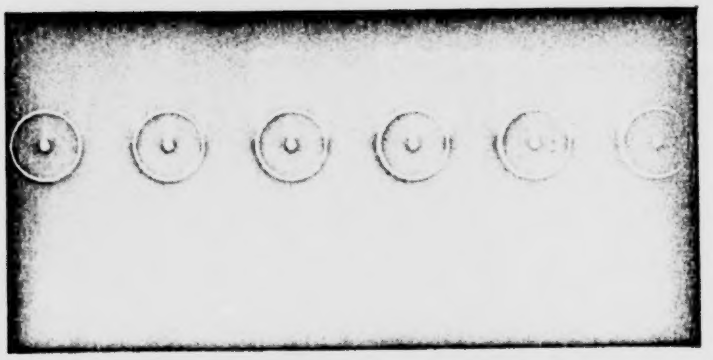

Fig. 1.1c The motion of an air disk with constant velocity on its horizontal surface. The disk moves from left to right. The light was flashed every 0.1 second to record the picture. This motion closely approximates the ideal motion of an isolated body with no external forces. The disk moves along a nearly straight line, and over nearly equal distances in equal time intervals. (Photo courtesy of Dr. Harold Daw, Seattle Physics Writing Conference.)

air disks placed anywhere on the flat table remain at rest? By the way, can you think of another way to level it?

When the air disk is released with a certain velocity (Fig. 1.1c) it continues with that same velocity (speed and direction), until it reaches the edge of the table. Check this for yourself from the figure. With dividers you can compare the set of successive displacements in equal time intervals. Can you invent a way of specifying the maximum deviation from straight-line motion? How accurately would you say this experiment verified the law of inertia? Did the speed remain constant to within ten percent? One percent? Were the deviations from straight-line motion significant?

Like any other single test the one illustrated in Fig. 1.1 verifies the law of inertia within a certain accuracy within a limited range of conditions. Variations can and should be made. The experiment can be repeated with different disks in different horizontal directions. The experiments can be performed in other laboratories and with other materials.

For higher speeds, where appreciable distances are covered in times too short for gravity to have any great effect, it is possible to check the law of inertia easily in three dimensions. Laboratory experiments with neutrons or other subatomic particles usually assume the validity of the law of inertia for the velocity in three dimensions. For example, they may require an isolated particle to pass through a set of small holes, lined up along a straight path, and arrive at a target at a certain time. The success of such experiments indirectly supports the validity of the law of inertia used in their design.

The law of inertia has been checked directly and indirectly in many situations. To within the accuracy expected, in every case where the data has been carefully examined, the predictions of the law of inertia have been verified. Here is a delightful simplicity in the motion of matter, and it applies to any isolated object: large or small, simple or complex.[1]

The law of inertia does not tell everything about the motion of an isolated object - only that so long as it is isolated it moves with constant velocity. As we now describe this motion in the language of vector mathematics we shall sharpen somewhat the distinction between what aspects are described by the law of inertia and what aspects are not.

When the object itself is very small compared with the experimental measurements of distance, we can consider the object itself as a mass point whose description consists of its location relative to some origin (three coordinate values). Then because of the relatively large scale of our distance measurements we can safely ignore any internal jiggling or tumbling of the object itself. Such a case is illustrated by the measurement in Fig. 1.2, in which the velocity of a tumbling cube is determined from its displacement during a certain time interval. The velocity $\vec{v}$, obtained from the positions of one particular point on the cube at two different times, differs slightly from the velocity that would result from following the motion of a different point. This uncertainty is not important if it is less than the accuracy with which we need to know $\vec{v}$. It is in this sense that we often describe the motion of a finite object in terms of the motion of an idealized particle, a mass with no extension, characterized entirely by the position of a point. The law of inertia predicts that a particle in isolation moves with constant velocity.

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[1] The first part of the film Inertia by E. M. Purcell (PSSC film #0302), could well be used here to reinforce this material in a "film clip" package, although a two-dimensional analysis would be preferable.

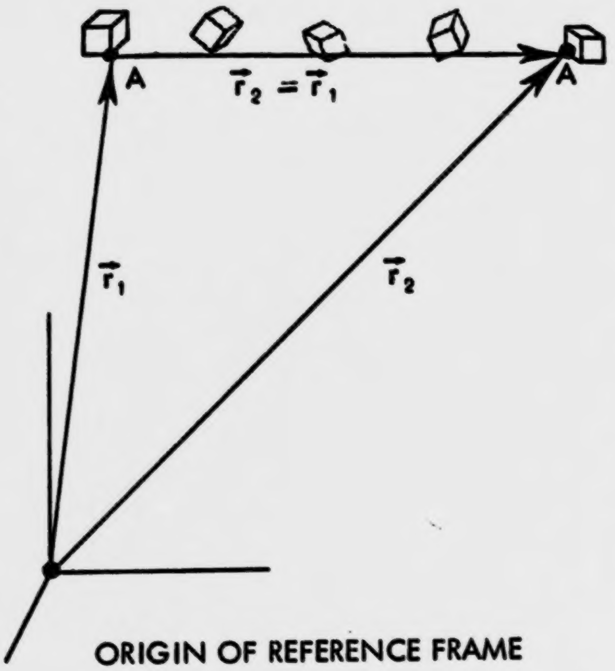


Fig. 1.2 The tumbling cube, free from external forces, carries the corner A from $\vec{r}_1$ at time t_1 to $\vec{r}_2$ at time t_2, through a displacement $(\vec{r}_2 - \vec{r}_1)$. The average velocity of point A during this interval of time is

$$\vec{v} = \frac{(\vec{r}_2 - \vec{r}_1)}{(t_2 - t_1)}.$$

The displacement $(\vec{r}_2 - \vec{r}_1)$ is taken as typical of the whole block because it is so much longer than any dimension of the cube, and $\vec{v}$ is taken for the "velocity of the cube." Following another point on the cube would result in a slightly different value for the velocity, but we assume that this uncertainty is less than the accuracy with which we need to know $\vec{v}$.

In actual laboratory work it may not be convenient to measure over very large distances relative to the size of the object. In many cases, however, we can find objects which do not have internal motions large enough to disturb our measurements of the whole body. For example, the cube in Fig. 1.3 is not tumbling. No matter what point on this cube is used for measurement of $\vec{v}$, the result is the same.

We shall see later that even for objects undergoing very complicated internal motions there is one well-defined point, the center-of-mass point, that continues to move with constant velocity if the object is isolated from external influences. For this point the law of inertia strictly applies. After we have discussed the law of conservation of momentum, you will be able to calculate the location of the center of mass for any ob-

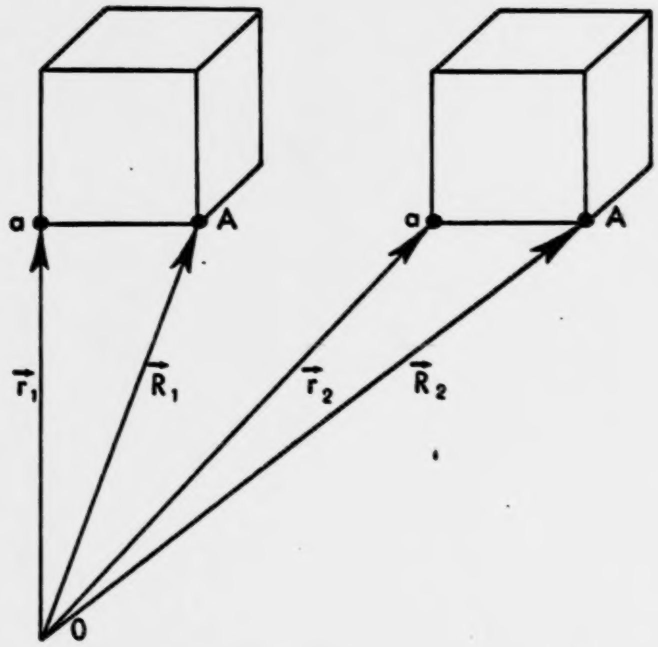


Fig. 1.3 When the cube is not tumbling, the two displacements $(\vec{r}_2 - \vec{r}_1)$ and $(\vec{R}_2 - \vec{R}_1)$ between times t_1 and t_2 are equal. Hence the velocity can be calculated from the displacement of point a or point A, or for that matter from any other point fixed on the cube:

$$\vec{v} = \frac{(\vec{r}_2 - \vec{r}_1)}{(t_2 - t_1)} = \frac{(\vec{R}_2 - \vec{R}_1)}{(t_2 - t_1)}.$$

ject. But you will probably not be surprised to learn now that the center of mass of an object with obvious symmetry (for example, a cube), corresponds to its geometric center (see Figs. 1.4 and 1.5).

1.4 A SIMPLE PRINCIPLE OF RELATIVITY

It is possible to show that the law of inertia obeys a simple principle of relativity. If the law is valid in one laboratory, then it will be true in the same form in any other laboratory that moves with constant velocity relative to the first.

Let us examine an experiment, using very simple equipment, as it takes place in two different laboratories. There is a device that one can make from a piece of string and a weight to demonstrate the sluggishness of matter. An elegant form of such a device is the plumb bob used by carpenters and surveyors to establish a vertical direction. When the bob or weight is suspended from the string at rest, it hangs vertically beneath the string support. The string is parallel to those of all other plumb bobs in the neighborhood.(Fig. 1.6a) The vertical string indicates that there is no horizontal force on the plumb bob at rest. This is in accord with the law of inertia, which predicts that a vertical string (no horizontal force), should be associated with a plumb bob at rest with regard to horizontal motion.

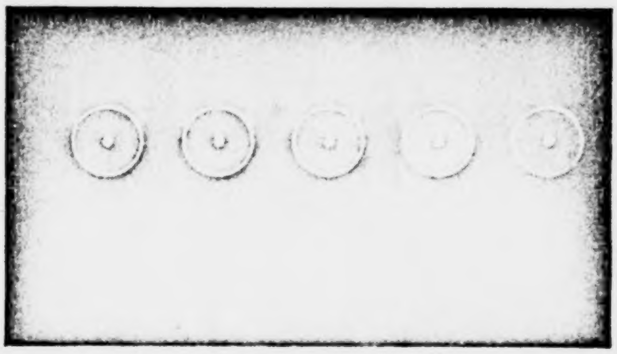


Fig. 1.4. A multiple-flash photograph of a cylindrically symmetric air disk which rotates about its center (marked with a cross) while the center moves with constant velocity. The law of inertia describes the horizontal motion of the center of the air disk, even when it is rotating. The disk enters the picture from the left, and the interval between flashes is 0.1 second. (Photo courtesy of Dr. Harold Daw, Seattle Physics Writing Conference.)

When the bob is put into motion by moving the string's support point (Fig. 1.6b), the string must be tilted to change the motion of the bob from rest to a finite velocity. The tilting string indicates that the bob is no longer isolated from horizontal forces. The tilt indicates the bob's inertia, or resistance to changing its velocity.

If the support then continues to move with constant velocity, the bob will eventually hang directly below it again (after the swinging has died down so that it has a steady position). As long as the support moves with constant velocity, the bob can move along at the same velocity with essentially no tilt to the string (see Fig. 1.6c). Even the tiny tilt that remains be-

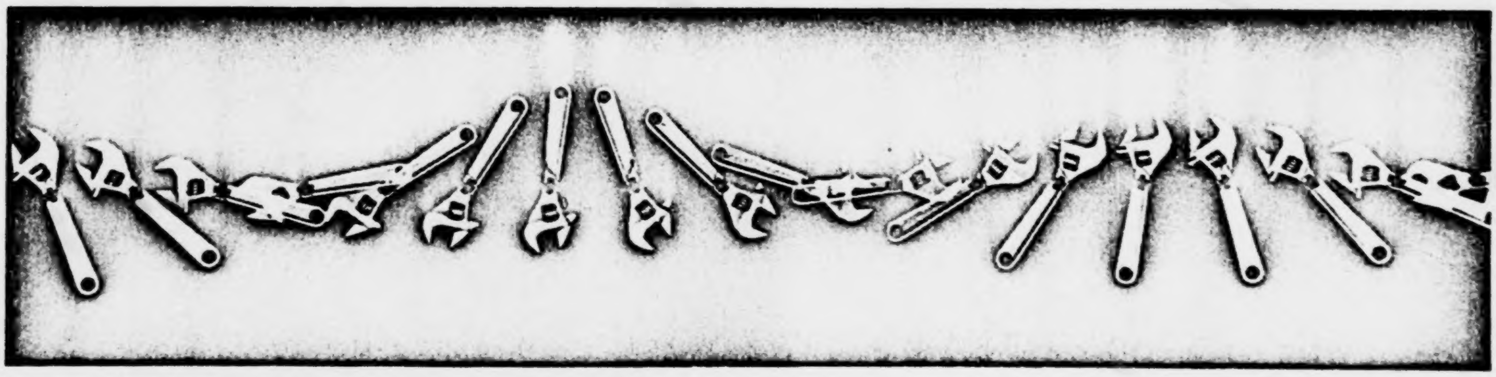


Fig. 1.5 A moving wrench photographed at 1/30 second intervals. The black cross marks the center of mass. (From PSSC Physics [D. C. Heath and Company, 1960].)

cause of air resistance can be removed by performing the experiment inside a closed automobile on a straight, level road. A bob suspended from a support moving along inside a closed car with constant velocity hangs vertically, beneath its support. And this is what the law of inertia predicts. The vertical string, implying a lack of any horizontal force, is associated with the constant velocity of the bob.

Now let us look at these results from two different viewpoints.

The Bonneville Salt Flats Experiment

The Bonneville Salt Flats, Utah, are chosen as the site of this experiment because they provide a large, flat horizontal surface on which an automobile can be driven with constant velocity.

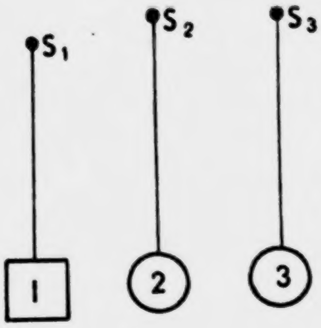
Joe, at rest on the ground at the salt flats, sets up a number of plumb bobs. To him they appear at rest and vertical.

Don, inside a car driving at a constant velocity of 50 miles per hour west, also sets up a number of plumb bobs fastened inside the car. To Don his own plumb bobs appear at rest and vertical. Therefore both Joe and Don can agree on the law of inertia for objects at rest. But note that they are able to apply the same law in two different laboratories, one of which moves with constant velocity relative to the other.

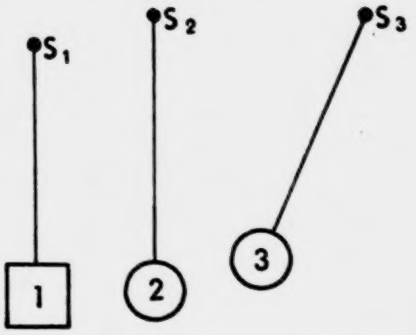
Joe, observing Don's plumb bobs moving at the constant velocity of 50 mph west, and still hanging vertical, confirms the law of inertia for objects in motion with constant velocity.

Meanwhile, Don has been looking at Joe's plumb bobs. Relative to Don's reference frame in the car, Joe's plumb bobs move at the constant velocity 50 miles per hour east, and hang vertically. Don can also verify the law of inertia for objects moving at constant velocity.

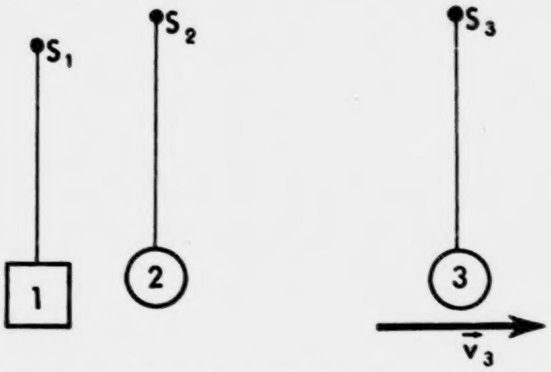
Whenever the car moves with velocity $\vec{v}$, Joe on the ground can verify the law of inertia for Don's plumb bobs traveling at velocity $\vec{v}$ and for his own at rest. But at the same time Don in the car can verify the law of inertia, relative to his own reference frame, for Joe's plumb bobs traveling at velocity $-\vec{v}$, as well as for his own at rest.



(a) Bobs 1, 2, 3 and their supports S_1, S_2, S_3 are all at rest. The two similar plumb bobs (2 and 3) as well as all others at rest (1) hang parallel.



(b) Bob 3 is being given a velocity to the right by moving S_3 to the right from its position of rest. The tilting string on bob 3 indicates the resistance to the change from rest to motion as the bob is accelerated.



(c) Bobs 1 and 2 and their supports S_1 and S_2 are at rest. Bob 3 and its support S_3 are moving to the right with constant velocity $\vec{v}_3$. Bob 3 rides directly under its support S_3, hanging parallel to bobs 1 and 2.

Fig. 1.6 A plumb-bob experiment.

It should now be clear that by such experiments two observers could never tell which one was moving and which one was at rest, for the law of inertia takes identical forms in both laboratories. That is the essence of this simplest principle of relativity, which is true for the law of inertia and for every other physical law for which it has been tested:

> The laws of physics are the same in two laboratories whose motions differ only by a constant relative velocity.[2]

You will see how closely the law of inertia and the principle of relativity are intertwined if we try to predict some of the results of the Bonneville Salt Flats experiment from two assumptions:

(1) The law of inertia *for objects at rest only*, and

(2) the principle of relativity. Joe, at rest on the ground, verifies as before with his plumb bobs the assumption (1), that objects at rest and in isolation remain at rest.

By applying the principle of relativity, Joe can predict that Don will obtain the same results in his laboratory (moving with constant velocity). Therefore, Joe can predict that he (Joe) will see Don's plumb bobs hanging vertically while moving at constant velocity. From these two assumptions he can predict that the law of inertia for *moving* objects will be verified. If it were not in fact verified by Don's experiment, it would be necessary to give up or somehow modify one of the two assumptions.

Notice that Don can apply the same reasoning and predict from his vertical stationary plumb bobs inside the moving car that Joe's plumb bobs, at rest on the ground, will also be vertical. If the law of inertia is correct for zero velocity, and if the principle of relativity is correct for the law of inertia, then the law holds true for every velocity.

1.5 THE THEORY OF GALILEAN RELATIVITY

The theory of Galilean relativity consists of two parts: the principle of relativity discussed in the previous section, and a recipe for translating the description of moving objects from the viewpoint of one laboratory to that of another moving with constant relative velocity.

During the discussion of the Bonneville Salt Flats experiment in the preceding section, we assume that the stationary observer (Joe), could look into the window of the automobile, passing with constant velocity, and "see" that the moving bobs were hanging vertically. If Joe were to describe a more complicated experiment going on in the moving car, one involving velocities as well as positions, it would require a little more attention to detail to relate Joe's description to that of Don inside the car. Don and Joe use different reference frames, moving relative to each other, to describe the same phenomena.

A *transformation* is a recipe for translating the descriptions of phenomena from the language of one reference frame to that of another. The *Galilean transformation*, in particular, translates the description of a moving object from the mathematical language of one reference frame to that of another moving at constant velocity relative to the first.

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[2] The famous theory of special relativity state by Einstein early in the twentieth century includes this principle as one of its postulates. However, it also considers very carefully the problems involved in describing moving systems when information can travel no faster than the speed of light (another of its postulates). Very briefly, the results of these considerations are that two observers moving relative to each other will not agree on such measurements as the length of an object, the duration of a time interval between two events, etc. Because these effects are small when the speeds are small compared with the speed of light (3×10^8 m/s) we shall not need to consider them in any of our own experiments.

1.5.1 The Galilean Transformation.

The essence of the Galilean transformation is this: Joe, at rest in a reference frame with origin 0, describes the motion of an object. At the time the object is at point P, described by the position vector $\vec{r}$ relative to the origin 0. Don, at rest in another reference frame with origin 0', describes the location of the object at the same point P with the position vector $\vec{r}'$, which gives the displacement from 0' to P, as in Fig. 1.7a. If 0' moves at constant velocity $\vec{u}$ relative to 0, and if we assume that Joe and Don measure time with clocks that run in perfect synchronism, then the object's velocity $\vec{v}$ relative to 0 (Joe), and $\vec{v}'$ relative to 0' (Don), are related by

$$\vec{v}' = \vec{v} - \vec{u}.$$

All our measurements of velocity, changes of velocity, etc., are based ultimately on measurements of position and time. Figure 1.7a shows how the position of one object P is described from two different reference frames (in two laboratories), at the same time. We assume that one observer, Joe, always describes things from the 0 reference frame and the other observer, Don, from the 0' frame.

When the origin of the 0' frame is separated from the origin of the 0 frame by the displacement $\vec{R}$ (Fig. 1.7a), the position of an object at point P relative to the 0 frame ($\vec{r}$) and its position relative to the 0' frame ($\vec{r}'$) are related by

$$\vec{r} = \vec{R} + \vec{r}'. \qquad (1.1)$$

This relationship should be clear from the geometry of Fig. 1.7a and the definition of vector addition. subtracting the vector $\vec{R}$ from both sides of Eq. (1.1) we obtain the equivalent statement,

$$\vec{r}' = \vec{r} - \vec{R}. \qquad (1.2)$$

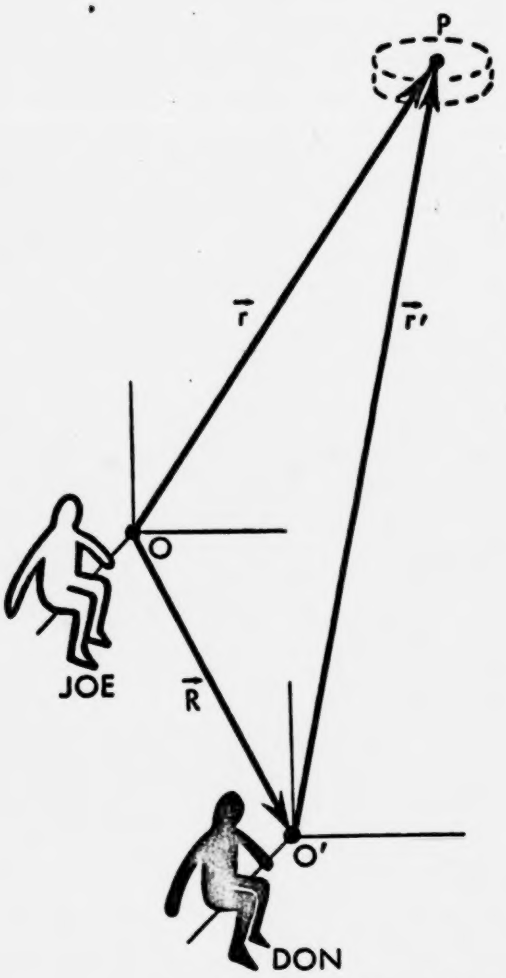

Fig. 1.7a An object at P, described from two different reference frames 0 (Joe's) and 0' (Don's), when the two origins are separated by the displacement $\vec{R}$, from the origin 0 to the origin 0'. The position $\vec{r}$ of P relative to 0 is related to its position $\vec{r}'$ relative to 0' by ordinary vector addition:

$$\vec{r} = \vec{R} + \vec{r}'.$$

Equation (1.2) can also be obtained directly by looking at the transformation from Don's (0') viewpoint as in Fig. 1.7b.

When the object is in motion, Joe and Don separately establish its velocity by position and time measurements. At his time t = 0, Joe finds the object at P_0, position $\vec{r}_0$ relative to 0; at time t_1 he finds it at position $\vec{r}_1$, as in Fig. 1.8. By the definition of average velocity $\vec{v}$, the displacement of the object from time 0 to time t_1 is given by

$$\vec{r}_1 - \vec{r}_0 = \vec{v}t_1. \qquad (1.3)$$

Now suppose that, during the time

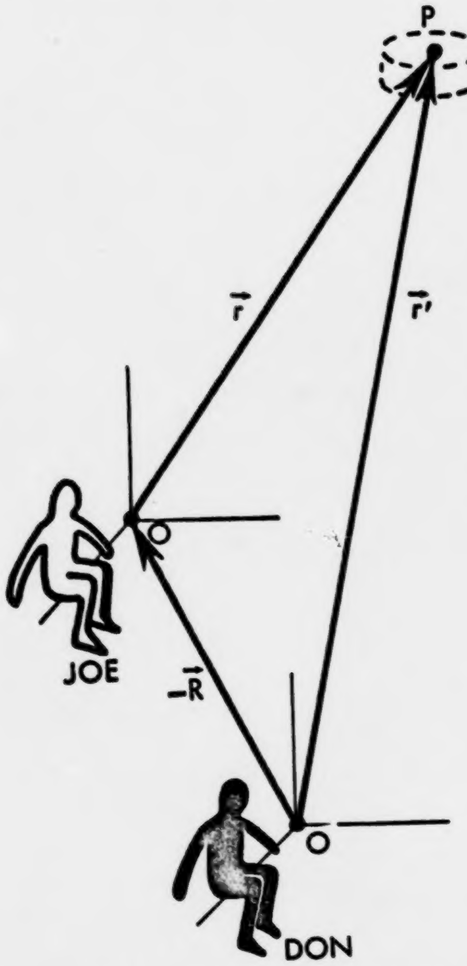

Fig. 1.7b This figure is equivalent to Fig. 1.7a, but it is drawn from Don's (O') viewpoint. From the geometry and the definition of vector addition

$$\vec{r}\,' = -\vec{R} + \vec{r}.$$

interval 0 to t_1, Don and his reference frame O' move relative to Joe with constant velocity $\vec{u}$. The origin O' will undergo a displacement $\vec{u}t_1$ during this time interval, as in Fig. 1.8.

From the geometrical relationships of the figures, we shall be able to describe the motion of the same object in the mathematical language of Don's frame of reference. Because Don moves relative to Joe, he will describe the positions and velocities of the object with numbers different from Joe's. However, there is a geometrical relationship between the two sets of measurements, a relationship which depends upon the relative velocity $\vec{u}$.

In Fig. 1.9 we see Don's two measurements of the positions of the object at points P_0 and P_1, the same points chosen by Joe. From the geometry of the figure you should see that

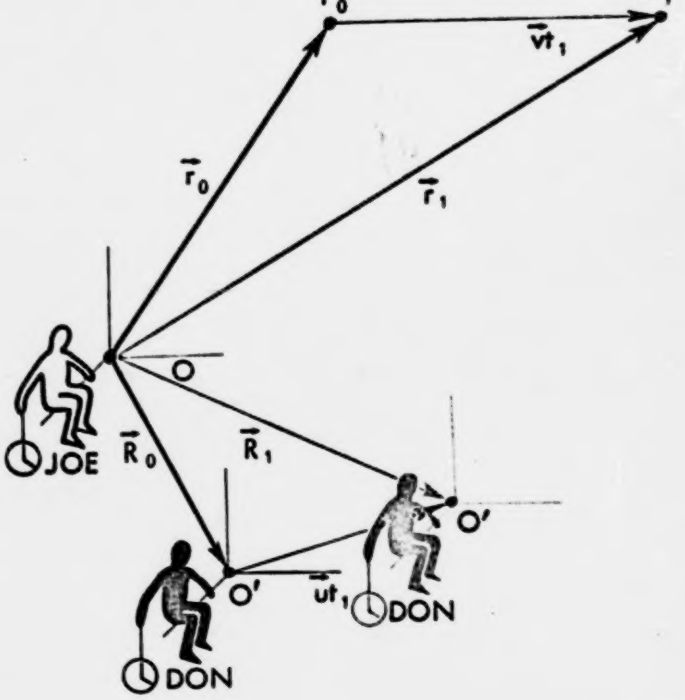

Fig. 1.8 The motion of the object from P_0 to P_1 and of the origin O' as seen from Joe's (O) frame of reference during the time interval 0 to t_1. If the object P moves with average velocity $\vec{v}$ relative to O,

$$\vec{r}_1 = \vec{r}_0 + \vec{v}t_1.$$

If the origin O' moves with constant velocity $\vec{u}$ relative to O,

$$\vec{R}_1 = \vec{R}_0 + \vec{u}t_1.$$

$$\vec{r}_1 = \vec{R}_0 + \vec{u}t_1 + \vec{r}_1\,', \qquad (1.4)$$

whereas $\vec{r}_0$ is given by

$$\vec{r}_0 = \vec{R}_0 + \vec{r}_0\,'. \qquad (1.5)$$

The length of the displacement $\vec{v}t_1$ between P_0 and P_1 in Joe's reference frame is obtained by substituting Eqs. (1.4) and (1.5) for $\vec{r}_1$ and $\vec{r}_0$ in Eq. (1.3):

$$\vec{v}t_1 = \vec{r}_1 - \vec{r}_0; \qquad (1.3)$$

$$\vec{v}t_1 = (\vec{R}_0 + \vec{u}t_1 + \vec{r}_1\,') - (\vec{R}_0 + \vec{r}_0\,'). \qquad (1.6)$$

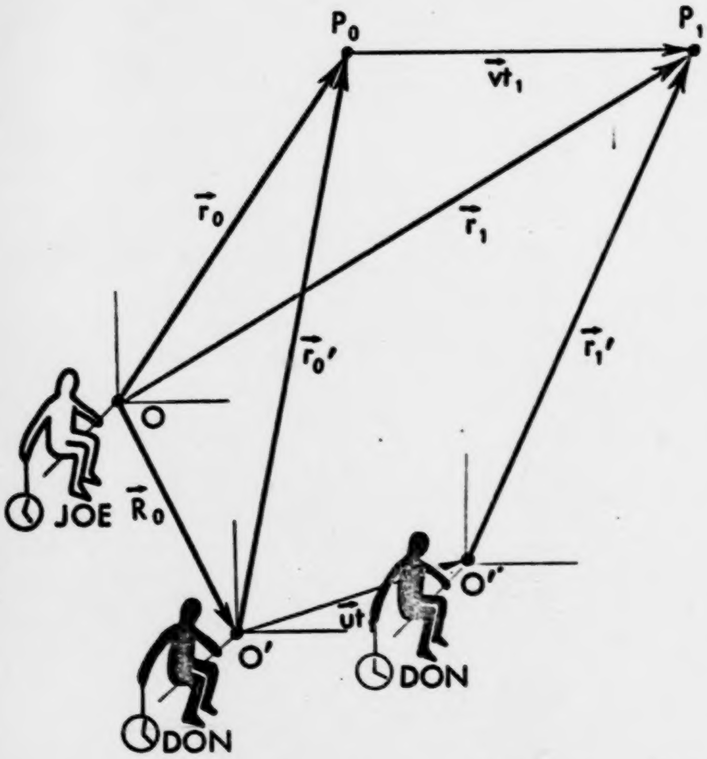

Fig. 1.9 The two measurements by Don ($\vec{r}_0'$ and $\vec{r}_1'$) of the positions of the object at points P_0 and P_1 relative to his own O' frame. The picture is drawn as though Joe were standing still and Don moving. From the geometry

$$\vec{r}_1 = \vec{R}_0 + \vec{u}t_1 + \vec{r}_1',$$

and

$$\vec{r}_0 = \vec{R}_0 + \vec{r}_0'.$$

Equation (1.6) can be simplified by noticing that the vector $\vec{R}_0$ disappears in the subtraction:

$$\vec{v}t_1 = \vec{u}t_1 + (\vec{r}_1' - \vec{r}_0'). \qquad (1.7)$$

Subtracting the vector $\vec{u}t_1$ from both sides of Eq. (1.7) and factoring the time t_1 gives a form which will be useful later,

$$(\vec{v} - \vec{u})t_1 = (\vec{r}_1' - \vec{r}_0'). \qquad (1.8)$$

Now we are ready to consider Don's velocity measurements. For simplicity, assume that Don agrees, with Joe, to start his clock at t = 0 when the object is at P_0. According to Don's clock the object reaches P_1 at time t_1'. By definition of average velocity, Don's measurement gives a value of

$$\vec{v}' = \frac{\vec{r}_1' - \vec{r}_0'}{t_1'}, \qquad (1.9)$$

for the average velocity $\vec{v}'$ in his frame of reference. Inserting the value of $\vec{r}_1' - \vec{r}_0'$ from Eq. (1.8) into Eq. (1.9) gives for the average velocity in the O' system

$$\vec{v}' = (\vec{v} - \vec{u})\ t_1/t_1'. \qquad (1.10)$$

We assume that the clocks of Joe and Don run in perfect synchronism;[3] i.e., that $t_1 = t_1'$. Then we obtain the simple vector equation,

$$\vec{v}' = \vec{v} - \vec{u}. \qquad (1.11)$$

The simple velocity transformation of Eq. (1.11) can be stated in words as:

> The velocity relative to O' is equal to the velocity relative to O minus the velocity of O' relative to O.

The complete details of the geometrical transformation that we have discussed, between the description of motion in one reference frame and that in another moving at constant relative velocity, are summarized in equations of the Galilean transformation, Eqs. (1.12).

O frame	O' frame
$\vec{R}_0$ = position of O' relative to O at t = 0 = t'; $\vec{u}$ = constant velocity of O' relative to O.	
$\vec{r}$ = position vector t = time $\vec{v}$ = velocity vector	$\vec{r}'$ = position vector t' = time $\vec{v}'$ = velocity vector

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[3] There is actually no universal time scale common to clocks moving with different velocities. For relative speeds, small compared with the speed of light, two observers will find that their clocks agree to within a very small correction. For laboratory velocities of macroscopic systems, this correction is negligible. If the relative speed ($\vec{u}$) were comparable to the speed of light (3×10^8 meters per second), the transformation equations of the special theory of relativity would have to be used instead of the Galilean transformation.

$$t' = t \qquad (1.12a)$$

$$\vec{r}\,' = (\vec{r} - \vec{u}t) - \vec{R}_0 \qquad (1.12b)$$

$$\vec{v}\,' = \vec{v} - \vec{u} \qquad (1.12c)$$

These equations in vector form are the equations of a general Galilean transformation to the O' frame from the O frame of reference. They depend on the constant velocity $\vec{u}$. Equation (1.12c) applies to the average velocity $\vec{v}$ measured over any time interval. Hence the equation can also be used to relate two instantaneous velocities $\vec{v}$ and $\vec{v}\,'$, determined in the limit of an arbitrarily small time interval.

In some cases we shall be able to use a much simpler form of the transformation. If the two origins O and O' coincide at $t = 0 = t'$, and if the relative velocity $\vec{u}$ is directed along the parallel x and x' axes,

$$t' = t \qquad (1.13a)$$

$$x' = x - ut,\ y' = y,\ z' = z \qquad (1.13b)$$

$$v_x' = v_x - u,\ v_y' = v_y,\ v_z' = v_z. \qquad (1.13c)$$

The transformations we have discussed take descriptions in the mathematical language of Joe's system and translate them into the quantities which describe moving objects in Don's moving system. Of course, Eq. (1.12) can be solved for $\vec{t}$, $\vec{r}$, and $\vec{v}$ if the reverse transformation is needed:

$$t = t', \qquad (1.14a)$$

$$\vec{r} = (\vec{r}\,' + \vec{u}t) + \vec{R}_0, \qquad (1.14b)$$

and

$$\vec{v} = \vec{v}\,' + \vec{u}. \qquad (1.14c)$$

Another way of obtaining these same equations is to go back to the beginning and redo all figures and analysis from Don's frame of reference. From this O' frame the origin O of Joe's frame moves with velocity $-\vec{u}$. From this O' frame the object "really" undergoes a displacement $\vec{v}\,'t_1$. Figure 1.10 shows an analysis of the situation as though Don were standing still and Joe were moving. Compare Fig. 1.9. The relationship among the displacements $\vec{u}t_1$, $\vec{v}t_1$, and $\vec{v}\,'t_1$ in Figs. 1.9 and 1.10 is illustrated in Fig. 1.11. The velocity diagram in the same figure illustrates the similarity of the vector Eq. (1.14c).

1.5.2 Galilean Relativity and the Law of Inertia.

The law of inertia is one of the simplest examples of a law of physics that takes the same form in two reference frames whose relative motion is one of constant velocity.

Our experiments have indicated that the law of inertia holds to a good approximation for horizontal mo-

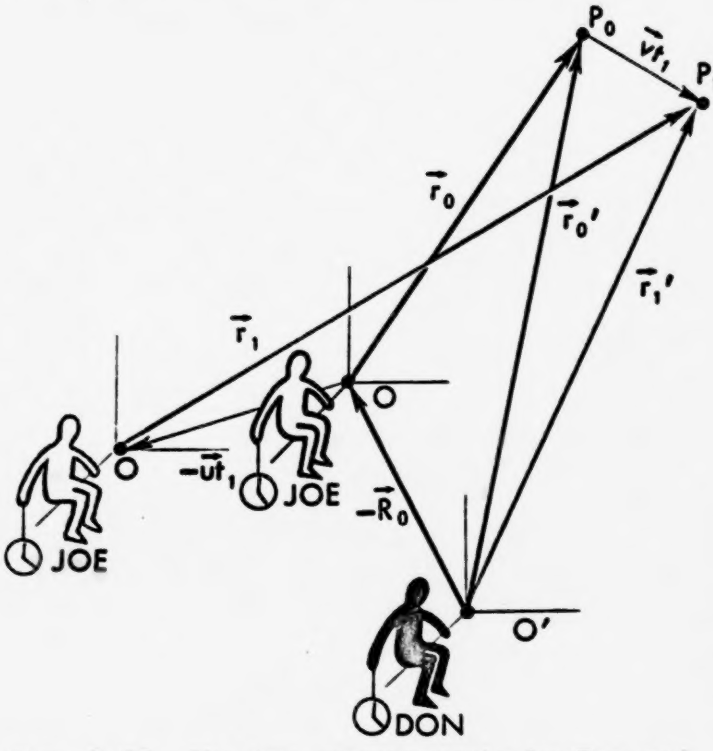

Fig. 1.10 The two measurements by Don and Joe of the positions of the object at points P_0 and P_1, drawn as though Don were standing still. Otherwise this picture represents the same events as Fig. 1.9, with which it should be compared. From the geometry

$$\vec{r}_1\,' = -\vec{R}_0 - \vec{u}t_1 + \vec{r}_1,$$

and

$$\vec{r}_0\,' = -\vec{R}_0 + \vec{r}_0.$$

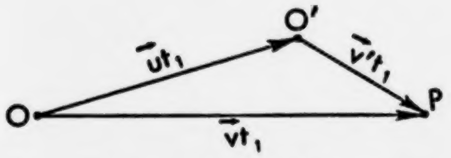

Fig. 1.11a The relationship among the three displacements: $\vec{u}t_1$, $\vec{v}t_1$, and $\vec{v}'t_1$ of Figs. 1.9 and 1.10. The points labeled O, O', and P signify that $\vec{u}t_1$ is the change in position of O' relative to O, $\vec{v}t_1$ is that of P relative to O, and $\vec{v}'t_1$ is that of P relative to O'. These displacements all occur during the time interval 0 to t_1.

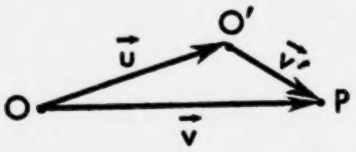

Fig. 1.11b The relationship among the three velocity vectors: $\vec{u}$, $\vec{v}$, $\vec{v}'$ of Figs. 1.9 and 1.10. The points labeled O, O', and P signify that $\vec{u}$ is the velocity of O' relative to O, $\vec{v}$ is that of P relative to O, and $\vec{v}'$ is that of P relative to O'.

tions at the surface of the earth, relative to a reference frame fixed to the earth. Let us assume for the purpose of the argument that we have found one reference frame (say far out in space), in which it holds exactly for motions in all three dimensions. The law of inertia states that an isolated mass moving with velocity $\vec{v}$ (relative to this frame), continues to move with the constant velocity $\vec{v}$ as long as it remains isolated. The name given to such a frame of reference is an inertial frame.

Let us imagine that an isolated mass m moves with constant velocity $\vec{v}$ relative to the inertial frame O, as in Fig. 1.12. The Galilean transformation tells how the motion of the same mass is described relative to another frame O', which moves with constant velocity $\vec{u}$ relative to O. We have seen in the first part of this section that relative to O' the mass m moves with velocity given by Eq. (1.12c),

$$\vec{v}' = \vec{v} - \vec{u}.$$

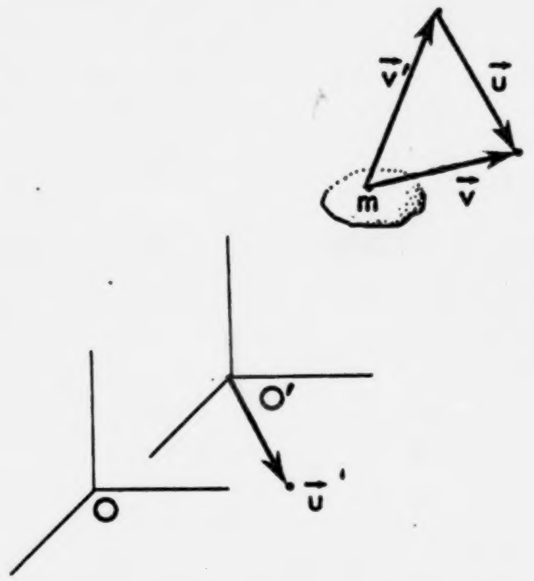

Fig. 1.12 The mass m, moving with velocity $\vec{v}$ relative to the reference frame O, moves with velocity $\vec{v}'$ relative to O'. If the frame O' moves with velocity $\vec{u}$ relative to O, then

$$\vec{v}' = \vec{v} - \vec{u}.$$

If the mass m is isolated, and if O is an inertial frame, $\vec{v}$ is a constant velocity. If $\vec{u}$ is constant, then so is $\vec{v}'$, and O' is also an inertial system.

By definition of the reference frame O', $\vec{u}$ is a constant velocity. Because O is an inertial frame, $\vec{v}$ is also a constant. Therefore, $\vec{v}'$ is a constant. The isolated mass m moves with constant velocity relative to O'. Thus O' is an inertial frame, one in which the law of inertia is valid.

Every reference frame O' moving at constant velocity relative to an inertial frame O is also an inertial frame.

1.6 DEVIATIONS FROM THE LAW OF INERTIA

A careful scrutiny of real motions at the earth's surface reveals small deviations from the behavior predicted by the law of inertia. Although these effects are not large enough to show up in simple laboratory experiments like the ones we have discussed, they are important in some natural phenomena. These are all connected with the fact that a reference system attached to the earth is not entirely suitable for determining the laws of physics in their simplest and most general form.

1.6.1 Inertial Frames

The earth rotates on its axis, and a point fixed to the earth near New York City travels around a circle with a circumference of about 17,000 miles every 24 hours. The 700 mph speed itself does not cause any difficulties. We have seen that if the law of inertia is valid in one frame, it is also valid in another frame moving at constant velocity relative to the first. What causes deviations from the law of inertia in the reference frame fixed to the earth's surface is the fact that the velocity of the reference frame is slowly changing with time as the earth rotates. The laws of motion appear in quite different forms in two laboratories, one of which is rotating relative to the other. A glass of water resting on a rotating phonograph turntable behaves very differently from a similar glass of water resting on the floor.

Because the earth rotates very slowly (once a day), the effects of its rotation are not normally important in the laboratory. They are sometimes important in very large-scale natural phenomena involving distances comparable to the earth's diameter. The earth's rotation is responsible for the fact that winds in the northern hemisphere do not blow radially toward a low-pressure area but instead spiral toward and around it in a counterclockwise direction. The weather map in Fig. 1.13 shows such a circulation around a low-pressure area in the northern Great Lakes region of the United States, and another around an offshore low-pressure area in the Atlantic.

The deviations from the law of inertia to be expected by an observer whose velocity is not constant with respect to an inertial system are illustrated in Fig. 1.14. In Fig. 1.14a the observer and his laboratory (the circle), with its reference frame are at rest in the inertial system O, and the observer verifies the law of inertia for the particle moving in his own laboratory. When the laboratory moves with constant velocity as in Fig. 1.14b, his laboratory frame is still an inertial frame, although he does not measure the same constant velocity that would be measured in another inertial frame.

If the observer's frame of reference moves in one direction but with changing speed relative to an inertial frame, as in Fig. 1.14c, the law of inertia will not be valid in his laboratory frame.

If the observer's laboratory reference frame rotates, as in the uniform circular motion of Fig. 1.14d, the law of inertia will not be valid in his frame. This case illustrates the difficulties experienced at the surface of the earth, although the effects due to the earth's very slow rotation are small.

If either the size or the direction of the velocity of the observer are changing relative to an inertial system, experiments described relative to his frame will not verify the law of inertia for that frame.

Suppose that one were actually to verify the law of inertia for a mass M in a direct experiment. The experiment would have to be carried out in interstellar space, far from any disturbing influences that might be important. To set up a frame of reference in such a situation requires a marker to denote the origin and some way to keep track of distances, directions, and time. The origin of the reference system could be marked with a rock O sufficiently small that one hopes it doesn't itself affect the motion of the mass M. The law of inertia predicts that the masses O and M will then separately move with constant velocities relative to an inertial system. Hence it predicts that their relative motion will be one of constant velocity.

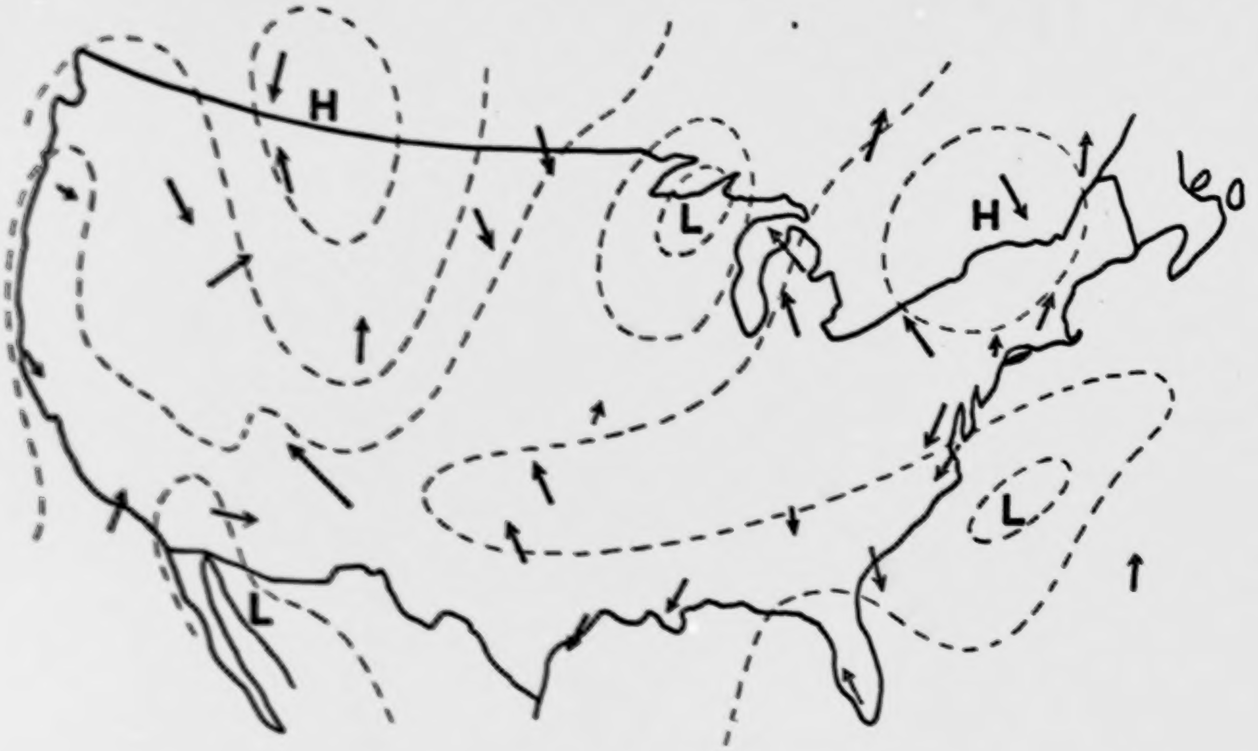

Fig. 1.13 This figure is a representation of the U.S. Weather Bureau weather map for July 31, 1965. The outline of the United States in black is overlaid with isobars (lines connecting points of equal atmospheric pressure) dashed in blue. Wind velocities are indicated approximately by the heavy blue vectors. The letters H and L identify areas of high and low pressure. The counterclockwise air circulations around the low-pressure areas centered in the northern Great Lakes and off the Atlantic coast are quite well developed.

Distance and time measurements alone cannot test for a constant relative velocity; some standard directions must be set up. The directions could be determined against the directions of "fixed stars" in the distance. To a high degree of accuracy the stars are so distant that their velocities and ours have almost no effect on the pattern of starry constellations during the course of a laboratory experiment. Hence they provide a satisfactory set of reference directions, if we make the reasonable assumption that the heavens do not share some uniform (and hence undetectable in the changes of star patterns) rotation which would make such a frame of reference system noninertial. Experiments carried out on the earth, in which corrections are made for its motions relative to the "fixed stars" indicate that they do indeed provide an accurate set of directions for setting up an inertial system.

QUESTIONS

1.1 How would you check whether the rock 0 at the origin had any effect on the results of the experiment?

1.2 How many directions need to be established to define a frame of reference? If you were able to measure only distances between reference rocks and the mass M, and could not see the stars, how many reference rocks would you need to test the prediction of a constant relative velocity between masses? Assume that all reference rocks could be placed at rest relative to each other.

1.6.2 Weightlessness.

In movies and television pictures, millions of eyes have seen what life

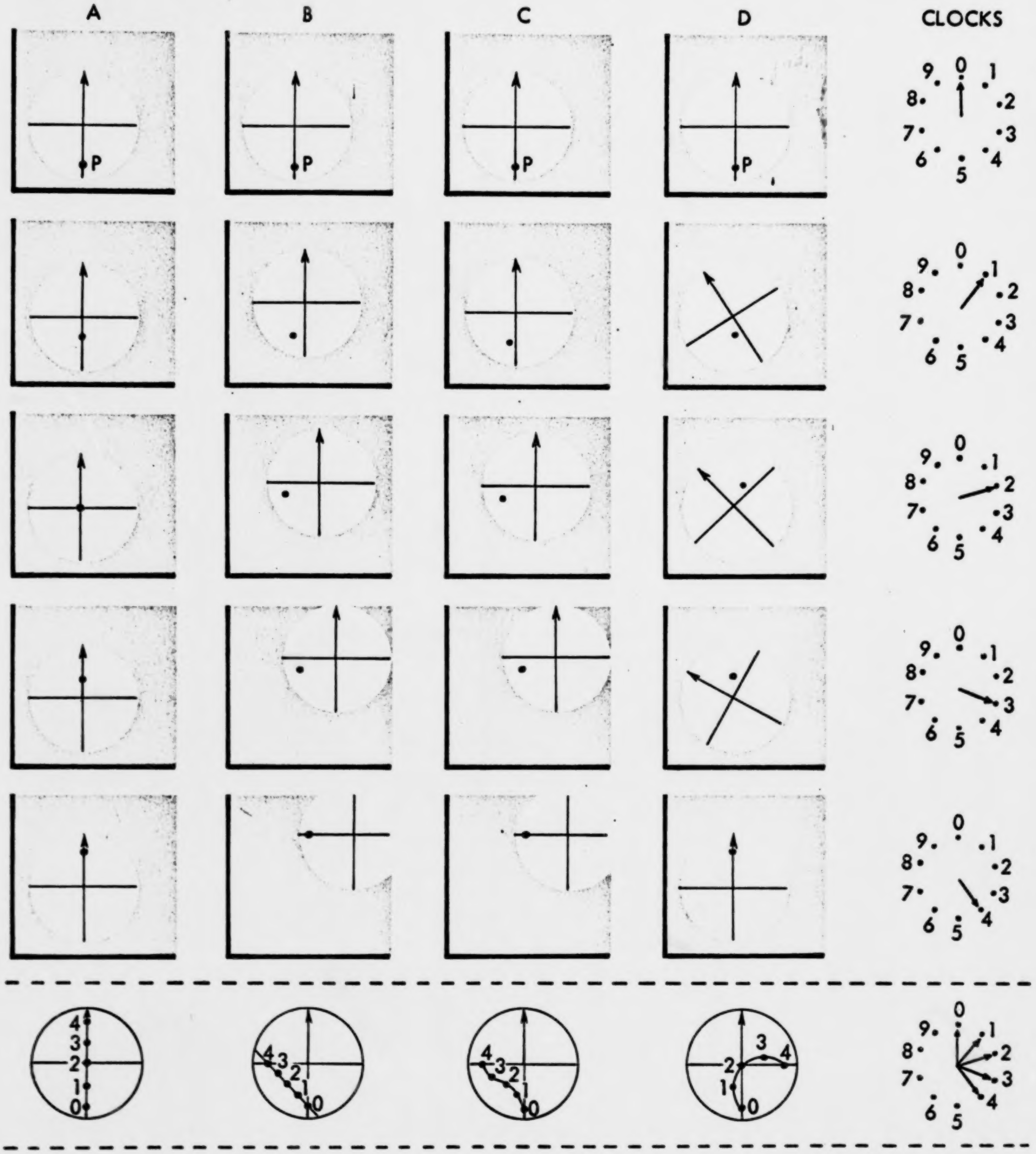

Fig. 1.14 The motion of a point mass P (or the center of mass of an extended object) in inertial and noninertial frames. An inertial frame is represented by the black frame in the lower left of each picture. The "clock" column indicates that successive horizontal rows show the position of P after successive equal time intervals. Read vertically downward, each column shows the same motion of point P, at constant velocity in the inertial frame at the lower left of the figure.

The white circle with the cross represents another frame of reference. Its motion is different in each column. In column A the circle is stationary and its frame is inertial. In column B the circle's frame is is like without gravity. Astronauts and the public looking over their shoulders have seen objects "floating in space" and moving in a straight line at constant velocity without falling, relative to an orbiting satellite of the earth.

It is amusing to see what is special about a reference frame fixed to an orbiting satellite. Objects which would "fall" relative to a reference frame fixed on the earth "float" relative to the satellite.

And yet there is a pull of the earth's gravity where the satellite is. We know that for certain, because that pull was used in the calculations that predicted the satellite's orbit. Briefly, the resolution of this paradox of weightlessness in the gravitational field of the earth is this. The satellite and its contents are falling together in an orbit around the earth. In the relative motion of the satellite and its contents the falling is not apparent.

Later we shall discuss how the combination of an initial horizontal velocity and a constant falling toward the earth's center can combine to produce a curved orbit around the earth. But first let us examine what other experiments tell us would happen in a falling laboratory.

Figure 1.15 shows two laboratories built of identical parts inside closed elevators near the earth's surface. Each has its own reference frame. Joe is in the elevator with the O frame and Don is in the elevator with the O' frame.

At a particular instant Bob cuts the cable on Don's elevator. Simultaneously Joe and Don let go of their hammers in surprise, Joe's hammer falls to hit his toe.

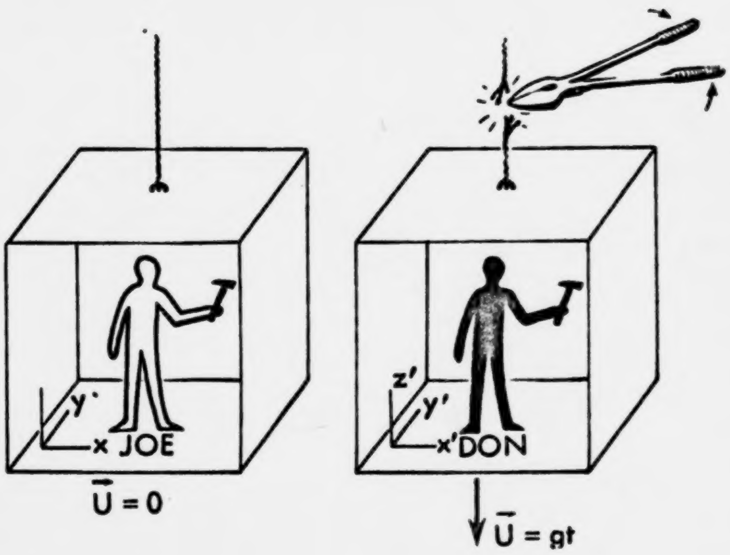

Fig. 1.15 Two elevator laboratories. Joe's elevator is at rest. Don's is accelerating at a constant rate after the cable is cut.

Relative to the earth-fixed O frame, Don's hammer also falls. But Don's elevator and Don fall, too. Now it is a remarkable thing about the gravitational pull of the earth that the observer, the elevator, and the hammer gain speed at the same rate. All objects falling at the same place under the influence of gravity gain equal amounts of vertical velocity in equal time intervals. Thus, relative to Don and his O' frame, the hammer does not fall (see Fig. 1.16).

The results of experiments with hammers, bullets, elevators, people, etc., at the earth's surface can be summarized as follows: If at the beginning of a time interval t the velocity of an object falling freely is $\vec{v}_0$, then at the end of that interval it is

$$\vec{v} = \vec{v}_0 + \vec{g}t, \qquad (1.15)$$

where $\vec{g}$ is a constant vector (9.8m/sec^2 down). If Don and Joe had thrown their hammers with initial

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an inertial one because it moves at constant velocity relative to the inertial frame at the corner of the figure. The motion of P at constant velocity relative to the circles of columns A and B is summarized in the superposition of points in the circle at the bottom of the column. The circle of column C is not an inertial system. It moves in one direction, but with a smoothly varying speed. The circle of column D, which rotates at a constant rate, is not an inertial system. The effects noted in column D are typical of those observed at the surface of the rotating earth.

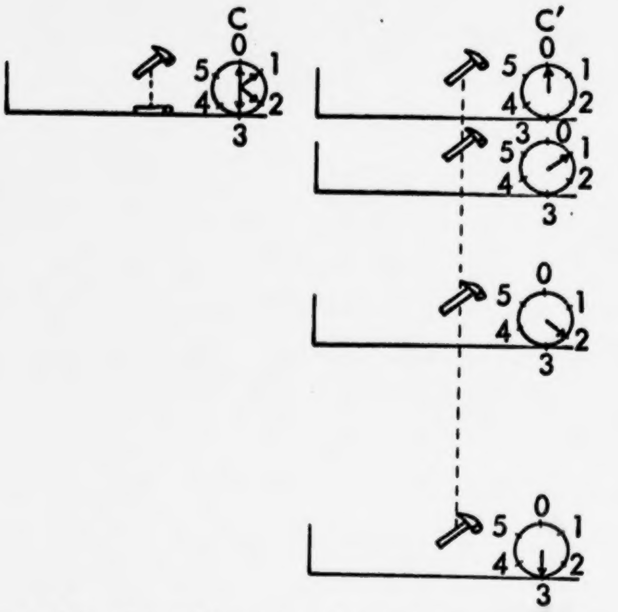

Fig. 1.16 Two freely falling hammers in a gravitational field. In the laboratory 0 at rest, the hammer falls to the floor. In the laboratory 0', which itself falls freely, the hammer appears to float. The clocks C and C' indicate successive equal time intervals.

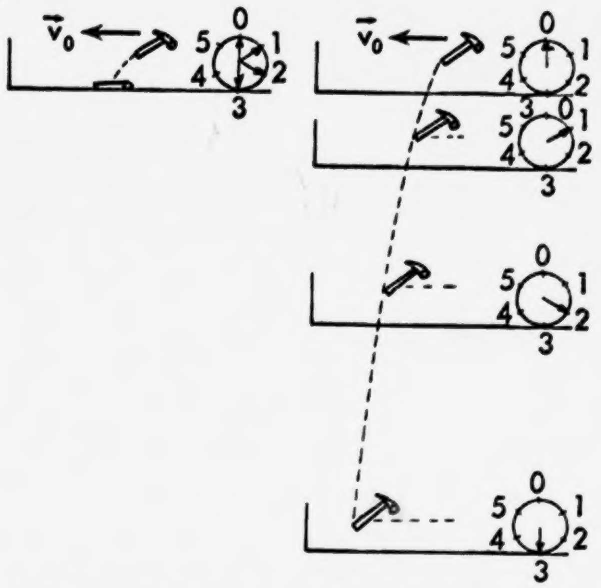

Fig. 1.17 Two hammers fall after having been given an initial velocity $\vec{v}_0$ to the left. In the 0 frame, at rest in the gravitational field of the earth, the hammer falls to the floor. In the 0' frame, freely falling with the hammer, the initial velocity $\vec{v}_0' = \vec{v}_0$ appears to continue unchanged.

velocity $\vec{v}_0$, then Joe's hammer would have described a curved path (actually, a part of a parabola), to the floor, as in Fig. 1.17.

But Don's hammer, moving with velocity $\vec{v}$ after a time t (Eq. 1.15), relative to the 0 frame, continues to move with velocity $\vec{v}_0$ relative to the 0' frame (see Fig. 1.17). Let us see how this comes about. Applying Eq. (1.15) to the velocity $\vec{U}$ of Don's elevator, we find that, starting from rest,

$$\vec{U} = \vec{g}t, \tag{1.16}$$

at time t later.

If the velocity of the hammer is $\vec{v}$ relative to the 0 frame, and the velocity of the 0' frame is $\vec{U}$ relative to the 0 frame, then the hammer's velocity relative to 0' is[4]

$$\vec{v}' = \vec{v} - \vec{U}. \tag{1.17}$$

But by substituting Eq. (1.16) for $\vec{U}$ and (1.15) for $\vec{v}$ into Eq. (1.17) for $\vec{v}'$ we obtain

$$\vec{v}' = (\vec{v}_0 + \vec{g}t) - \vec{g}t$$

$$\vec{v}' = \vec{v}_0. \tag{1.18}$$

The hammer moves with constant velocity relative to Don's freely falling elevator. If Don regards the hammer as isolated, he finds that his 0' frame is an inertial frame. A reference frame falling freely in the earth's uniform gravitational field is an inertial frame if other masses falling also under the influence of gravity are treated as isolated.[5]

A freely falling elevator would be, for a short while, a suitable place for demonstrating the law of inertia in three dimensions. The lack of weight would have a corollary benefit in reducing friction. Objects could move freely through the thin air rather than being dragged over a rough table. Experiments we now per-

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[4] The Eq. (1.16) was developed in section 1.5 for two reference frames moving with constant velocity. It can also be applied when the velocity $\vec{U}$ is changing if it is applied to instantaneous velocities.

[5] The apparent absence of gravity in a properly accelerated frame of reference is very important to the theory of general relativity, which deals with accelerated frames and gravitation.

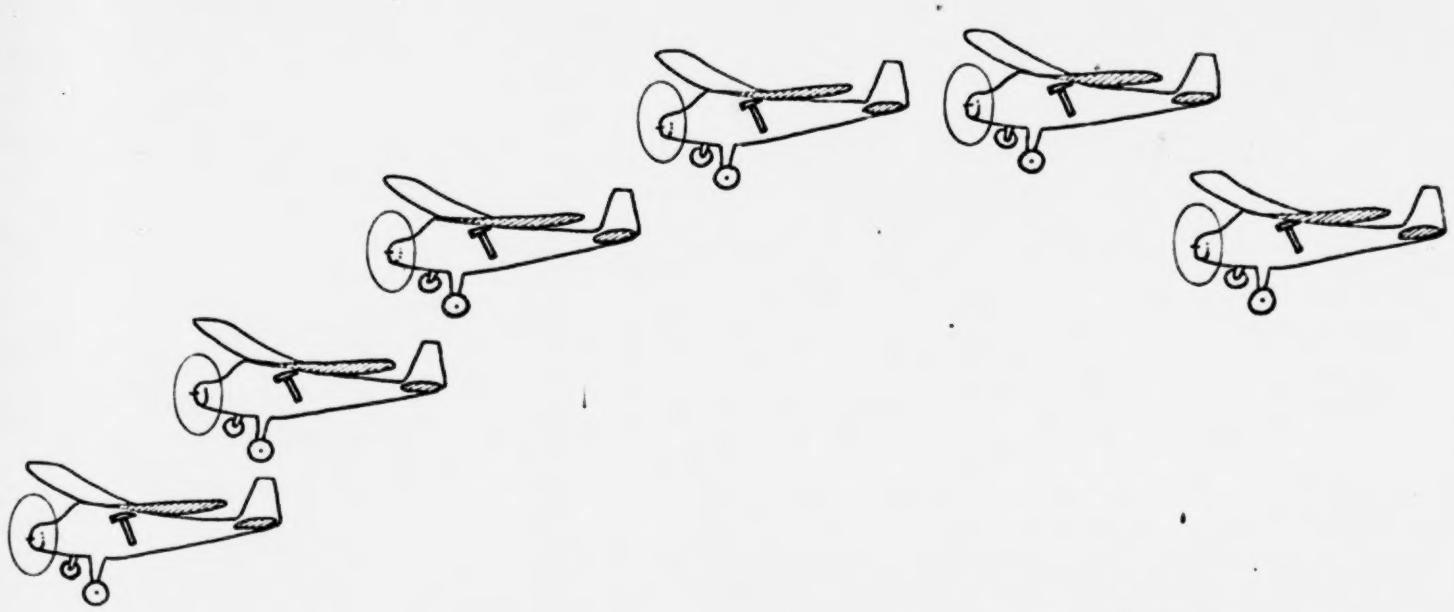

Fig. 1.18 An airplane flying in a parabolic path, in which during each time interval Δt the velocity gains an amount $\vec{g}\Delta t$ in the downward direction. For such a path the airplane provides a weightless reference frame in which the hammer inside "floats." To provide an inertial frame the airplane would have to fly without changing its attitude, as in the figure. In actual fact, an airplane would not fly that way very well.

form with great effort using air disks, etc., could be done easily with hammers, people, china cups, almost anything.

To acquaint astronauts with the sensations of weightlessness, an ingenious method for producing weightlessness without the dangers of falling elevators has been devised. An airplane with initial velocity $\vec{U}_0$ can be flown in a (parabolic) path such that for 20 seconds or so the velocity changes in the vertical direction in the same manner as in free fall:

$$\vec{U} = \vec{U}_0 + \vec{g}t. \qquad (1.19)$$

Such a path is illustrated in Fig. 1.18. Relative to the airplane, objects within it do not fall when released. The airplane provides a weightless frame. An object released at time zero within the plane has velocity $\vec{v}_0'$ relative to the plane and velocity

$$\vec{v}_0 = \vec{v}_0' + \vec{U}_0 \qquad (1.20)$$

relative to the ground. At a later time t its velocity relative to the ground is given by Eq. (1.15) while that of the plane is given by Eq. (1.19) above. Relative to the airplane the object has velocity

$$\begin{aligned} \vec{v}' &= \vec{v} - \vec{U} \\ &= (\vec{v}_0' + \vec{U}_0 + \vec{g}t) - (\vec{U}_0 + \vec{g}t) \\ &= \vec{v}_0'. \end{aligned} \qquad (1.21)$$

The airplane in this special trajectory is an inertial frame. The situation is illustrated in Fig. 1.18.

A satellite in an orbit circling the earth has a motion similar to that of the airplane in a parabolic path. It has acceleration toward the center of the earth.

Although the "weightlessness" effect is much more general, let us restrict ourselves to a discussion of a simple circular orbit within a few hundred miles of the earth for which the gravitational effect of the earth does not change much in size. The size $|\vec{g}|$ of the constant acceleration given to all masses is about the same as it is at the surface

$$|\vec{g}| = 9.8 \text{ m/sec}^2. \qquad (1.22)$$

Of course the direction of $\vec{g}$ is at each point on the orbit given by the local direction to the earth's center (geocentric direction).

You should recall from your study of the description of motion that a circular motion at constant speed involves an acceleration which is constant in size and directed toward the circle's center. The size of this acceleration is

$$|\vec{a}| = \frac{v^2}{r}, \qquad (1.23)$$

where v is the constant speed around a circle of radius r.

The gravitational influence of the earth near its surface gives all freely falling objects, including the satellite and its contents, an acceleration given by Eq. (1.22) in the geocentric direction. Thus, if the tangential velocity is exactly correct for the radius r, the satellite and its contents continue to move in a circular orbit with constant speed, the acceleration being provided by the influence of the earth's gravity. The correct speed for such a circular orbit can be obtained by setting $|\vec{a}|$ of Eq. (1.23) equal to $|\vec{g}|$. Thus,

$$v^2 = r\,|\vec{g}|,$$

or

$$v = \sqrt{r\,|\vec{g}|}. \qquad (1.24)$$

For a radius of about 4000 miles, this gives a speed of about 18,000 mph.[6]

An object isolated except for the effects of gravity and at rest relative to the satellite will continue with the satellite in the circular orbit, both undergoing the same acceleration of size, $|\vec{g}|$ toward the earth's center.

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[6] If the speed is not exactly that of Eq. (1.24) the orbit is not circular. The satellite and objects within are still equally affected by gravity, and a situation of weightlessness prevails, but it is more complicated to describe.

2 THE LAW OF MOMENTUM CONSERVATION

2.1 MOMENTUM AND THE CONSERVATION LAW

One very important aspect of linear motion of an object is that it has linear or translational momentum. The momentum of an object equals the product of the mass of the object and its velocity. Since velocity is a vector quantity and mass is a scalar, momentum is a vector quantity. The momentum is written as $\vec{p}$,

$$\vec{p} = m\vec{v}.$$

Momentum and the conservation law for momentum are the main topics of this chapter. It is the purpose of this chapter to develop the concepts of momentum, of its conservation law, and of mass itself.

Momentum is basic to the structure of physics. The law of conservation of momentum is one of the unifying principles of physics.

What do we mean when we say that something is conserved? Let us try to define this briefly now, with the expectation that full appreciation of conservation laws comes only with use and understanding of the real thing. If the quantity conserved is momentum, then we state, "For any isolated system, the total momentum of that system is constant." The conservation law is independent of how complicated the processes may be going on within the system, whether they involve living or nonliving things, expansions or contractions, fast or slow changes. If the total amount of momentum of the system is known at any one time, and if the system remains isolated, then the total momentum will have that same value at any other time.

Once firmly established, such a general statement as the conservation law can be an extremely powerful tool. A conjecture made concerning the system can be declared possible or absolutely impossible depending on whether or not the conjecture agrees with the conservation law.

As we will see, direct application of conservation laws can bring new insight to the physics in the system.

The law of momentum conservation is basic and so nearly elementary that it can be considered early in the development of mechanics. Of the two alternatives - postulating the law and then deriving and testing its consequences or developing the law from a series of experiments - we have chosen the latter. Some of the experiments will be thought experiments that can readily be visualized, and some can be actually performed in your classroom. Of course, many more than we can actually perform were needed to give momentum conservation the strong position it holds today in physical theory, but our experiments will be representative of the important points in the development of the law.

The first situations we study will be simple ones with which we can easily cope. From these we shall be able to perceive the simplicity in more complex situations and eventually to analyze some of them in detail.

The first chapter dealt with motion of an isolated object. The object was not necessarily a point mass. Even though it may have consisted of several component parts, it was nevertheless visualized as one object, or a system "as a whole." In going from one, to more than one object, we will find that some of the basic concepts of physics, such as the mass of each object, momentum conservation of the system, and force between objects, can be developed by considering just two objects or particles interacting with one another. In this chapter, the concepts will usually be developed first in the specialized case which is

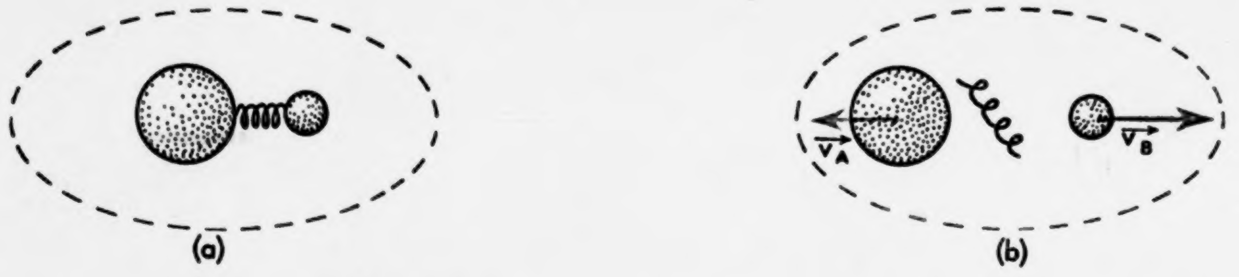


Fig. 2.1 The system of two spheres and a small spring. That the system is isolated is schematically indicated with an imaginary dotted line.
(a) The system is shown just before the small compressed spring is released.
(b) The system is shown after separation produced by release of the small spring. Vectors $\vec{v}_A$ and $\vec{v}_B$ are the velocities of A and B, respectively.

easier to visualize or to reproduce in the classroom. Subsequent to this, a brief general development will be given which will be more formal and also more abstract in that it will not be tied to a particular given physical experiment. All concepts carry over to the general from the special without faltering.

Let us consider this problem: There are two spheres "floating" together in outer space, isolated and far from the earth or other objects. One sphere, A, is massive and other other, B, has low mass. (Mass has been described in Chapter 1 as a sluggishness of matter, a property that expresses matter's inertness to changes in its velocity.) A small compressed spring located between A and B is suddenly released and forces A and B apart as shown in Fig. 2.1.

Perhaps the obvious question asked of the laws of physics should be, "Given a knowledge of the mass of A, the mass of B, and complete specification of the spring, with what speed will A be moving and with what speed will B be moving after the spring expansion?" That is a good question and it has an answer if all the information is available.

But consider other questions.

(1) Is there "a quantity" of this isolated system which is unchanged by (is conserved during) this spring expansion?

(2) Is there "a quantity" of motion equally acquired by A and B?

(3) If there are quantities of motion unequally acquired by A and B, what determines the division between A and B?

Perhaps you detect the qualitative difference between these questions and the "obvious" question above.

There are clear answers to (1) and (2) and from these answers, the "obvious" question as well as (3) are readily solved. The answers to questions (1) and (2) suggest general, universal conservation laws basic to all of physics. Two of these laws will be developed in the chapters which follow.

In our laboratory we will do experiments similar to the one described above except that they will not be performed "floating in outer space." Instead the objects A and B will move along a straight horizontal track. This is a special track known as an "air track" or "air trough," and the objects are glide blocks ("gliders"), that move freely on a film of air. The gliders are supported by many small jets of air continuously coming out of the track. The motion of each glider on its cushion of air is practically frictionless. Although friction is an interesting and complicated subject, we don't want to study it right now! Certainly friction is evident and useful in our lives for without it, for example, we couldn't take steps in walking. Some roughness between our feet and the sidewalk is needed for firm footing. In mechanics experiments, however, where we would like to find out what controls motion

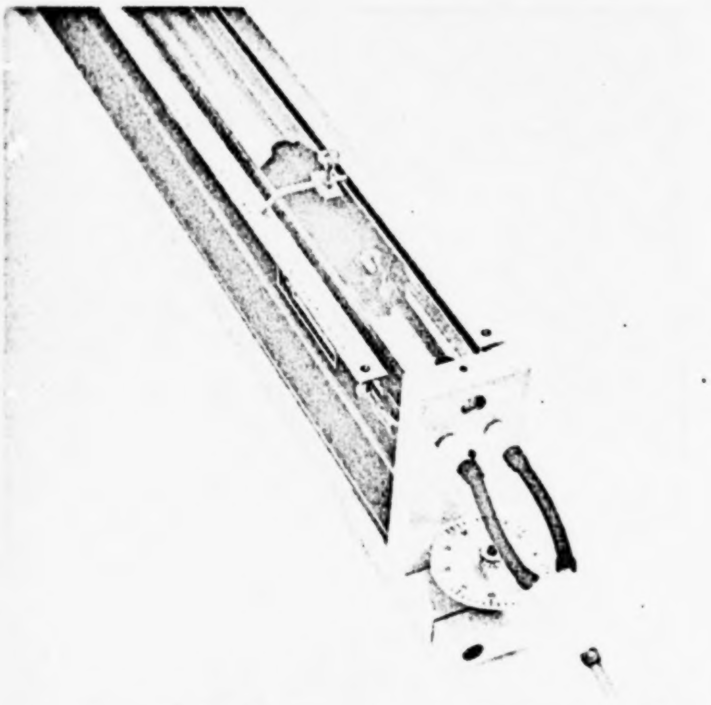
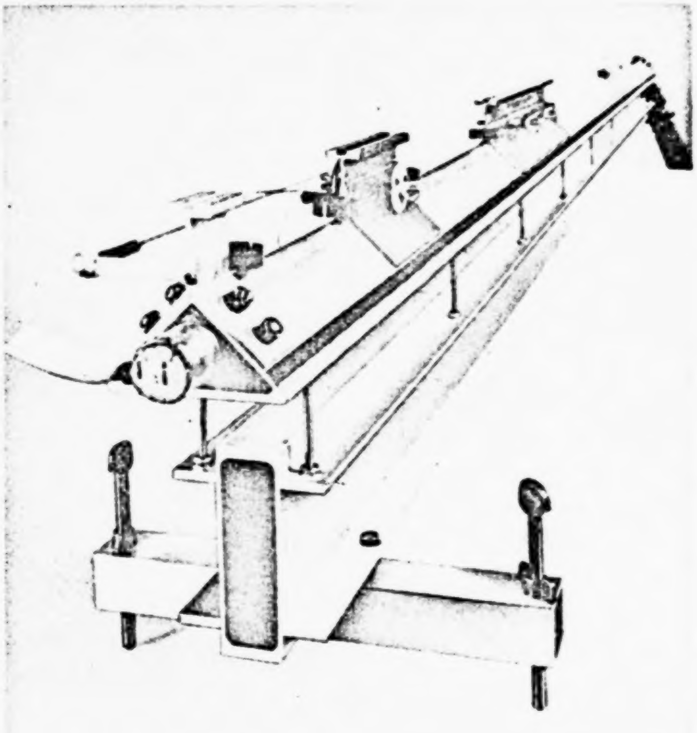


Fig. 2.2 (a) Photograph of an air-trough with a typical glider in position near the end of the trough. Both the glider and the end post are fitted with bumper springs. The hoses at the end admit air to the manifold (shown in Fig. 2.3). This glider is fitted with a marker to indicate its position on the scale that is mounted on the side. (Photograph of Search Linear Air Trough, courtesy of Macalaster Scientific Corporation.)



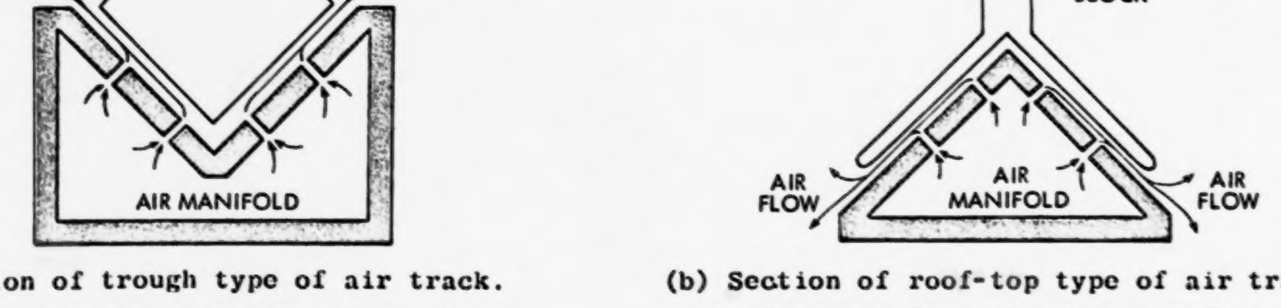
(b) Photograph of a roof-top type air track with two gliders in position on the track. Both of the gliders and the end post are equipped with bumper springs. The screws used to adjust the vertical height of the track can be seen near the end of the track. The air input hose is at the back end of the track, as seen here. The forward end of the manifold is closed so that the only escape for the air is through the vents along the track. (Photograph of Stull-Ealing Linear Air Track, courtesy of Ealing Corporation.)

of things, the friction between our object and whatever supports it would be a nuisance. Fortunately, it is almost eliminated by use of an air track.

2.2 AIR TRACK AND THE TWO-PARTICLE EXPLOSION

The air track is actually a hollow tube which can be shaped either as a trough or as a roof top. The two usual types of tracks are shown in Figs. 2.2 and 2.3.

When the track is in use, air



(a) Section of trough type of air track.

(b) Section of roof-top type of air track.

Fig. 2.3 Cut-away sections of the two types of air tracks. The air manifold is continuously supplied with air at a pressure higher than atmospheric pressure. The air escapes through all of the small air vents located along the upper surface of the manifold. This air flow, which is indicated in (a) and (b), forms the air cushion which supports the glider.

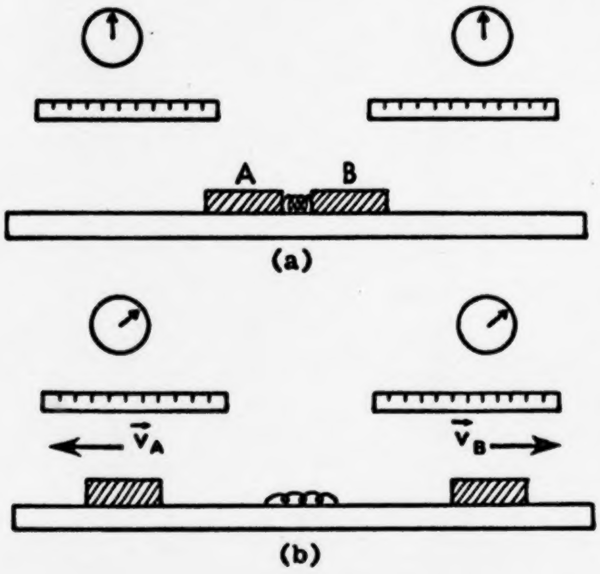


Fig. 2.4 Schematic representation of a symmetric air-track explosion experiment. (a) The glide blocks are shown in position just before the spring is released. (b) The blocks are shown in motion after the spring has been released.

above atmospheric pressure is pumped into the hollow interior and this air escapes continuously through the small jet vents (holes) that are located all along the track surface. For either track in Fig. 2.2, the glider is large enough to cover several vents and it rides or "hovers" over the track on the cushion of air provided by the jets. As long as the glide block does not go too fast, it glides along frictionlessly on its air support without scraping the track.

We are provided with a number of identical glide blocks made of, for example, aluminum. Being identical means being made of the same "stuff" and being made the same size and shape. Let us also assume that we are provided with meter sticks and stop watches so that the velocities of our gliders can be measured. Other, more sophisticated, timers such as strobe cameras, photoelectric cell timers, etc., may be available. Such refinements may be needed for high-speed carts, but they involve no change in the principle of measurement. All involve the determination of velocity from length and time measurements. Note that no knowledge of the laws of mechanics is needed to measure velocity, $\vec{v}$.

To study motion, something is needed (as the literal motivator) which will get our objects, the glide blocks, into motion in a controllable way. We will use a small coiled spring for this purpose. Many other devices, such as a chemical explosive (a toy "cap"), an electric motor, or compressed gas, could also be used. The small spring has advantages for us at this time, and it will be used first.

The spring will be compressed between the left, A, and the right, B, gliders shown in Fig. 2.4. After the system is quickly released, the spring will expand and force the gliders apart. The gliders will move along the track and the velocity of each can then be measured as it goes along.

The method of spring release deserves attention. One should not rely on using his hands for holding the gliders together and for releasing them. A person may be neither nimble nor quick enough to avoid disturbing the glider motion. A proven technique is to use a string loop which slips over a pin on each cart and holds them together. The string can be burned with a match with little or no disturbance to the gliders as they fly apart during the spring expansion.

The track should be level. This is readily accomplished by adjusting the track position while using the glider at rest as a level indicator. With the track horizontal, we can neglect all gravitational effects and with the air turned on, we very nearly simulate a "floating in space" experiment in one dimension along the track.

All of the spring "explosions" that we can observe on our track will be found to have two qualitative features in common:

(a) While the spring is expanding, it is always in contact with both gliders. Both gliders then lose contact with the spring simultaneously. (This means that the spring is small and that its motion can be neglected in the discussion that follows.)

(b) After the gliders lose contact with the spring, each moves with its own constant velocity along the track.

The observation (a) above is important in the development of mechanics.

The observation (b) follows directly from the law of inertia discussed in Chapter 1. The free glider continues to move with constant velocity because it is isolated on our frictionless track. The constant velocity implies that the instantaneous velocity of a glider right after an explosion has the same value as the average velocity we measure for any convenient length Δx of its trip along the track. It then follows that the instantaneous velocity of each glider after an "explosion" is found as the measurement of the average velocity of each glider. The average velocity is easier to measure since it equals

$$v_{x\ \text{ave}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1},$$

and Δx can be taken as a convenient displacement which allows reasonably convenient measurements of Δt. Question: Is the track location of the interval $(x_2 - x_1)$ at all critical? That is, can $(x_2 - x_1)$ be near the end of the track as in Fig. 2.5a, or should it be near the explosion as in Fig. 2.5b?

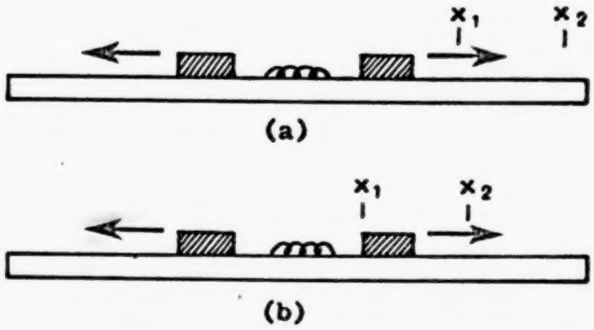


Fig. 2.5 (a) Location of interval $(x_2 - x_1)$ near end of track. (b) Location of interval $(x_2 - x_1)$ near explosion. The arrows represent velocity vectors.

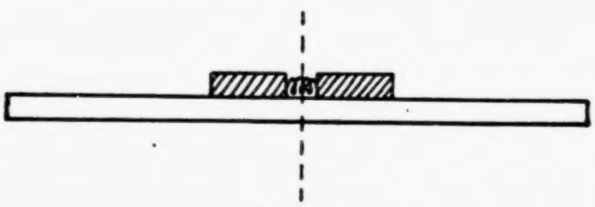


Fig. 2.6 Two identical gliders are shown in position before the spring is released. The dotted line runs through the center of the system.

2.3 AIR-TRACK EXPERIMENTS AND RESULTS

2.3.1 Experiment 1. Identical Left- and Right-Side Objects.

Two identical objects or gliders, A and B, are placed at the center of the air track as shown in Fig. 2.6, and "exploded" apart by the spring. After the explosion, the velocity of A, $\vec{v}_A$ and the velocity of B, $\vec{v}_B$ are measured. A number of small springs of various shapes and degrees of stiffness are used, and for each explosion $\vec{v}_A$ and $\vec{v}_B$ are measured.

If we were to examine the measurements, we would find that in each explosion, the speed acquired by A equals the speed acquired by B. Since $\vec{v}_A$ and $\vec{v}_B$ are vectors in opposite directions,

$$\vec{v}_A + \vec{v}_B = 0.$$

This is not a very surprising result since the experiment is perfectly symmetric about a vertical line drawn through the center of the spring. What is there to favor the left over the right? If A and B were interchanged, no basic experimental change would exist since A and B are identical aluminum blocks.

2.3.2 Experiment2. Nonsymmetric Explosion.

The left-right symmetry is removed by making the left glider larger or smaller than the right glider. This can first be done by simply using more than one glider and by connecting them rigidly together possibly as a train

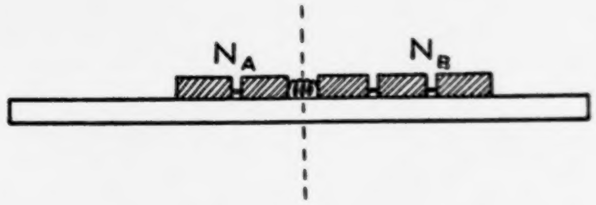


Fig. 2.7 Two trains of identical blocks, N_A and N_B, are shown in position on the air track before the spring is released.

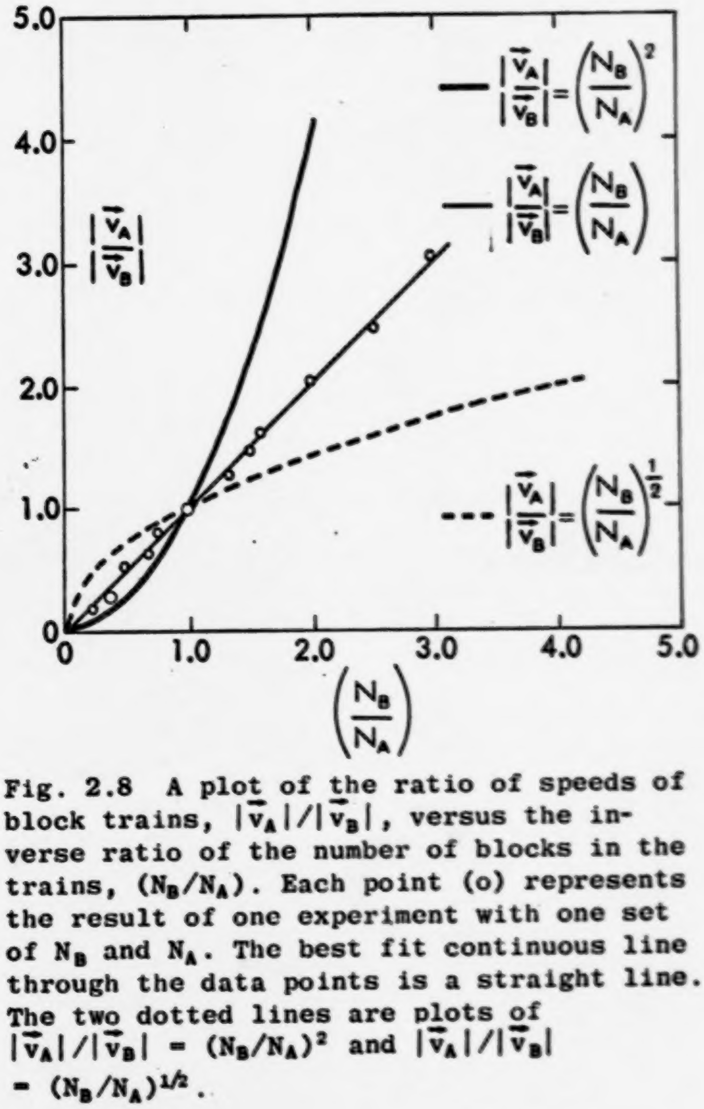


Fig. 2.8 A plot of the ratio of speeds of block trains, $|\vec{v}_A|/|\vec{v}_B|$, versus the inverse ratio of the number of blocks in the trains, (N_B/N_A). Each point (o) represents the result of one experiment with one set of N_B and N_A. The best fit continuous line through the data points is a straight line. The two dotted lines are plots of $|\vec{v}_A|/|\vec{v}_B| = (N_B/N_A)^2$ and $|\vec{v}_A|/|\vec{v}_B| = (N_B/N_A)^{1/2}$.

or as a stack of glide blocks. Let N_A and N_B be the number of identical glide blocks on the left and right sides of the spring, respectively, as in Fig. 2.7.

Let us try some explosions with various values of N_A and N_B. The speeds developed by A and B are found to be no longer equal. What we find is that the smaller the number of blocks on a side, the faster that side moves. From a study of the measurements, we would find that for many values of N_A and N_B and for many sizes and kinds of "explosion springs," the ratio of velocities varies in inverse linear proportion to the ratio of the number of identical glide blocks.

Or: the ratio of velocity magnitudes is

$$\frac{|\vec{v}_A|}{|\vec{v}_B|} = \frac{N_B}{N_A}. \tag{2.1}$$

Does the inverse proportionality surprise you? Take an extreme case with $N_B = 10$ and $N_A = 1$, and $N_B/N_A = 10$. Perhaps you can visualize large B slowly moved by the explosion and A scooting away rapidly from the explosion. Better yet, do the experiment!

How could we study these measurements? How could we establish the linear (first power) dependence of the ratio of $|\vec{v}_A|/|\vec{v}_B|$ upon (N_B/N_A) and not some other "power law" dependence such as, for example, $(N_B/N_A)^2$ or $(N_B/N_A)^{1/2}$? The qualitative dependence of $|\vec{v}_A|/|\vec{v}_B|$ on (N_B/N_A) may be suggested by visual observations of the experiment. If we were to suspect some dependence of $|\vec{v}_A|/|\vec{v}_B|$ upon (N_B/N_A), we might try plotting the results as ratios on a graph as in Fig. 2.8.

The plot demonstrates visually and quantitatively the linear dependence of $|\vec{v}_A|/|\vec{v}_B|$ upon (N_B/N_A) and clearly rules out the other power law dependences mentioned earlier.

The conclusion expressed by Eq. (2.1) says nothing of the size of each individual value of $|\vec{v}|$. Big springs can be observed to produce faster motion, and small springs produce slower motion for both A and B. With a particular small spring, for example, both A and B move slowly, but the ratio of $|\vec{v}_A|$ to $|\vec{v}_B|$ is still given by the ratio of N_B to N_A.

If we now rewrite the equality

$$\frac{|\vec{v}_A|}{|\vec{v}_B|} = \frac{N_B}{N_A}$$

as

$$N_A|\vec{v}_A| = N_B|\vec{v}_B|, \tag{2.2}$$

we can note that the left side of the Eq. (2.2) describes side A only and contains no dependence on $\vec{v}_B$. The right side similarly depends on B only. This equation tells us that there is still a left-right equality in the motion even though there are more blocks (and thus more "stuff" on one side than the other). The product of N and $|\vec{v}|$ still has a kind of left-right symmetry even though we thought we had removed this symmetry!

In this one-dimensional experiment, the vectors which represent velocities, $\vec{v}_A$ and $\vec{v}_B$, are opposite in direction. Since the velocity is a vector quantity, the product $N\vec{v}$ must also be a vector quantity because N is a scalar. The direction of $N\vec{v}$ is given by the direction of $\vec{v}$.

Since $N_A\vec{v}_A$ and $N_B\vec{v}_B$ have equal magnitudes, it follows that

$$N_A\vec{v}_A = -N_B\vec{v}_B. \qquad (2.3)$$

These vectors are shown in Fig. 2.9. Equation (2.3) can also be written as

$$N_A\vec{v}_A + N_B\vec{v}_B = 0. \qquad (2.4)$$

When the conclusion is stated as in Eq. (2.4), it appears as a conservation principle. In words, Eq. (2.4) says that the value of the sum of products of $N\vec{v}$ has a value after the explosion that equals the value of the sum of products $N\vec{v}$ before the explosion. The value of this sum is conserved, remains constant, at a value which for this experiment is zero. If Eq. (2.4) is valid, then no matter may be the nature of the explosion, small or weak, fast or slow, the sum of products of $N\vec{v}$ must be zero at the conclusion of the explosion. Equation (2.4) is written to emphasize the conservation concept and we certainly recognize limitations in its application at this time since it was only developed for a particular series of aluminum glide blocks on an air track!

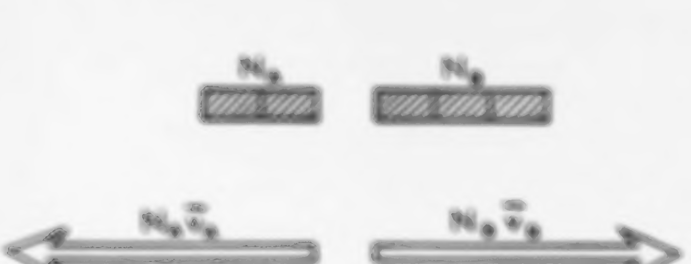


Fig. 2.9. Since N_B is greater than N_A, $|\vec{v}_B|$ is less than $|\vec{v}_A|$. The vectors $N_A\vec{v}_A$ and $N_B\vec{v}_B$ are equal in magnitude and opposite in direction.

2.4 THE CONCEPT OF MASS

To extend the experiment to other materials, we could make up a series of identical gliders of copper, gold, plastic, etc., and we would always find a conservation principle for the sum of $N\vec{v}$ for any one particular substance.

Would the law hold if we mixed up, say, copper and aluminun blocks? In order to do this, we need a way to compare copper to aluminum in the context of this mechanics experiment. When is a copper block equivalent to an aluminim block? The equivalence can be established experimentally by putting one aluminum glider on the left and one copper glider on the right. If the copper block acquires a greater speed than the aluminum block, then a larger copper glider will have to be chosen. If the copper block moves more slowly than the aluminum, then a little copper will have to be shaved from it until, with careful adjustment,

$$|\vec{v}_{\text{copper}}| = |\vec{v}_{\text{aluminum}}|, \qquad (2.5)$$

and we can say that the two blocks are equivalent in this experiment.

A comparison between a third material, Lucite plastic, for example, and aluminum could also be established. In a comparison of equivalent Lucite and aluminum blocks one would observe

$$|\vec{v}_{\text{Lucite}}| = |\vec{v}_{\text{aluminum}}|. \qquad (2.6)$$

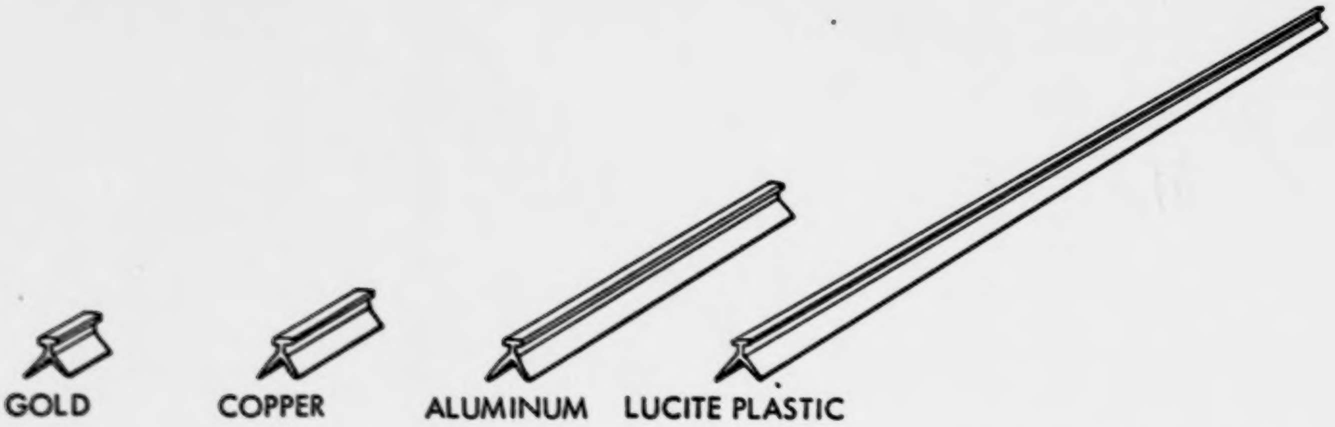


Fig. 2.10 A collection of glide blocks having the same cross section. Each block has been cut to be equivalent to the aluminum glider in air track explosion experiments. If the aluminum glider has a length of 1.00 unit, the length of the gold glider is 0.140 unit, the copper 0.303 unit, and the Lucite plastic, 2.29 units.

If we have established that for three separate gliders,

$$|\vec{v}_{\text{copper}}| = |\vec{v}_{\text{aluminum}}|$$

and

$$|\vec{v}_{\text{Lucite}}| = |\vec{v}_{\text{aluminum}}|,$$

can we now assert that $|\vec{v}_{\text{copper}}|$ will equal $|\vec{v}_{\text{Lucite}}|$ in "air-track explosion" experimental comparison between them? One might expect that the copper glider and the Lucite glider would be equivalent, but this would not be established until the gliders were actually compared in a similar "air-track explosion" experiment. Might not other untested factors affect the equivalence? Would color, shape, and metallic character have an effect? It turns out that these factors do not matter. If it is determined by air track explosions that for a certain material, "J"

$$|\vec{v}_J| = |\vec{v}_A|,$$

and for material "K"

$$|\vec{v}_K| = |\vec{v}_A|,$$

then experiment will show that

$$|\vec{v}_J| = |\vec{v}_K|.$$

Figure 2.10 shows a collection of gliders matched by experiment for equivalence.

In a continuation of the air-track explosion experiments, it can be demonstrated that equivalent blocks of the various materials can be freely substituted for one another and the conservation principle of Eq. (2.4) still sholds.

$$N_A\vec{v}_A + N_B\vec{v}_B = 0.$$

It can also be demonstrated that double blocks (twice as long) and fractional blocks can be used and the conservation law still holds provided the meaning of N is changed to allow for multiple blocks and fractions of a block.

Clearly, then N_A is a measure of the amount of stuff in the "A" group and N_B is a measure of the amount of stuff in the "B" group. From the results of these experiments, N is also a measure of the sluggishness or inertia of the train of blocks. Trains of glider blocks that have large N acquire low velocities in the experiment. We have used N to label the "amount of stuff" and the inertia of each train in this experiment because this particular experiment deals only with blocks. If we could melt the aluminum blocks of each train and form them into another shape (for example, a sphere), and repeat the explosion experiments "floating in outer space" the same velocities would be measured. Some characterization or label less restrictive than the "number of blocks" is needed for each train. There is such a physical quantity, and it is called mass.

The mass of a block or group of blocks is directly proportional to the number, N.

For sets of blocks A and B,

$$\frac{M_A}{M_B} = \frac{N_A}{N_B}. \qquad (2.7)$$

We say that the particular Lucite plastic block and the aluminum block of Fig. 2.10 have the same mass because they are equivalent in behavior in the air-track experiment. Both blocks contain the same amount of stuff or matter, although they are very different in other physical and chemical characteristics.

The similarity in the behavior is similarity in one-dimensional motion. There is no spinning or tumbling motion to the block as it slides along the track. The motion is purely translational. Certainly the blocks shown in Fig. 2.10 are not mechanically equivalent in all respects. The variations in lengths could be very important in some mechanical systems involving rotations of these objects, but in pure translational motion, the objects are equivalent.

Mass is more than a bare number. Mass has physical units or "dimensions," and the dimension we shall usually use is the kilogram. So far we are capable of measuring ratios of masses only with the air-track experiment. From the definition of mass given above, we determine the mass ratio of two sets of blocks from the velocity ratio, as measured in one of the air-track explosions.

$$\frac{M_A}{M_B} = \frac{|\vec{v}_B|}{|\vec{v}_A|} \qquad (2.8)$$

The air-track explosion experiments help in the development of the concept of mass as a measure of the inertia of an object or particle. In this work the mass of a block or glider is inversely related to its final speed on the track. Or, the bigger the mass of the glider, the greater the inertia of the glider. The larger the mass of an object, the less responsive it is to causes that would put it into motion or change its motion.[7] The mass is a measure of the inertia of the object.

The mass is also a measure of the amount of matter in an object. It is simply extensive in that if the volume of an object is increased by adding the same material in the same state, the mass increases in direct proportion to the amount of material.

Mass is a positive scalar quantity. Only one number is needed to specify the mass of an object. The connection between the scale of mass developed here and the unit of mass, the kilogram or the gram, can be established by experiment. A standard mass is placed on one of two identical gliders and then the usual speed measurements following another spring explosion are made. For example, if the empty gliders each have mass M_a, the mass of the standard is, say, 0.1000 kilogram, and the ratio of the speed of the empty glider to the speed of the glider carrying the standard is $|\vec{v}_a|/|\vec{v}_{sa}|$, then,

$$\frac{M_a + 0.1\ \text{kg}}{M_a} = \frac{|\vec{v}_a|}{|\vec{v}_{sa}|},$$

and

$$M_a = \frac{0.1\ \text{kg}}{\left[\frac{|\vec{v}_a|}{|\vec{v}_{sa}|} - 1\right]}.$$

The mass of each glider can then be so measured and each glider can be labeled with its own mass in kilograms. Once these glider blocks are so calibrated, they can be used to determine masses of other gliders in units of kilograms.

With the definition of mass which has been developed, the air-track equipment could be used as a mass measuring device. To measure an unknown mass, M_0, the unknown could be placed on a pan or in a compartment of one glider, M_1, so that the loaded mass of that glider would be $(M_1 + M_0)$. In an explosion experiment with another glider of mass M_2, the final speeds $|\vec{v}_1|$ and $|\vec{v}_2|$ of the loaded block (1) and the second block (2) would be observed. Then

---

[7] Whether an object is "put into motion" or has its motion changed is only a question of which frame of reference the motion is described from. This is more thoroughly discussed in Section 2.6.

$$\frac{(M_1 + M_0)}{M_2} = \frac{|\vec{v}_2|}{|\vec{v}_1|},$$

and finally

$$M_0 = M_2 \frac{|\vec{v}_2|}{|\vec{v}_1|} - M_1 .$$

This is admittedly an awkward way of measuring mass, and it would never be practical in a chemist's laboratory. The analytic balance used by chemists is certainly simpler, faster, and more accurate.

The determinations of the mass of a chunk of matter by the air-track method and the chemist's balance are in perfect agreement. There is, however, a difference in the principle of operation of these two methods. The air-track method of determining mass makes a comparison of the inertial properties of one chunk of matter with the inertial properties of a known mass. The determination of mass with the analytical balance makes a comparison of the gravitational attraction of the earth for one chunk of matter with the gravitational attraction of the earth for a known mass.

The development of a mass concept and the measurement of mass with the air track or similar equipment are completely independent of the earth's gravitational attraction. In fact, the horizontal air track was used to isolate the experiments on motion from the effects of gravity. The air-track experiments could be performed just as well in a space ship moving at constant velocity, millions of miles from any planet. Such a space ship would be free of gravitational effects and would provide an ideal inertial reference frame in which to work. Actually, the air-track would then not be needed as a gravity isolator!

There are many physical and biological processes in which the masses of the elements of the system are important and all gravitational effects are unimportant. It is satisfying to know that a mass concept and scale can be developed independent of gravitation.

Mass as a measure of the inertia of matter is a concept drawn from experiment. The validity of this concept depends upon its usefulness in understanding and analyzing more and more complex physics. It is a concept which has withstood such tests.

There are other concepts of mass. The famous law of universal gravitation discovered by Isaac Newton states that the force of gravity exerted by one mass on another is directly proportional to the product of their two masses and inversely proportional to the square of their separation distance. A concept of mass developed from this law requires some knowledge of "force." We will not discuss this further until after we develop the concept of "force." There is also the mass-energy equivalence contained in the special theory of relativity. This concept of mass will be discussed in the study of energy and energy conservation.

2.4.1 Dependence of Mass on Velocity.

If our experiments were repeated with more and more powerful springs, we would, within our ability to measure, obtain the same value for the mass of an object, no matter how fast the gliders could be separated. We would conclude that the mass is independent of velocity. This conclusion holds (that is, is accurate enough), at the low speeds within the reach of our air tracks and springs, but other experiments with microscopic particles (protons and electrons, for example), in the high-energy physics laboratories of the world have demonstrated that mass is not independent of speed. Appreciable changes in mass are not apparent unless the speed of the particle approaches the speed of light. The velocity dependence of mass is given by Einstein's special theory of relativity as

$$M = \frac{M_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}, \qquad (2.9)$$

where M_0 is the mass of the object at rest ($v = 0$), c is the speed of electromagnetic waves (light) in a vacuum, and v is the speed of the mass.

The Eq. (2.9) for the velocity dependence of mass says that a moving mass is less responsive to forces that would change its motion than the same object at rest. Experiments support the form of this equation. Equation (2.9) is a succinct statement of "how matter behaves" with no elaboration of how this comes about. Substituting numbers from our own air-track experiments shows that the correction of Eq. (2.9) to the one we use,

$$M = M_0,$$

is negligible within experimental error.

The speed of light is very large, 3×10^8 meters/second or 186,000 miles/second. With our air-track equipment, we could never make our gliders move at speeds approaching the speed of light. In fact, such equipment could not make any macroscopic object (that is, an object perceivable with the human senses), move at speeds approaching 186,000 miles/second. For example, the mass of an object moving at 3 meters/second (about 6 mph) is bigger than its rest mass by a factor of 1.00000000000000005. Even at speeds comparable to those of earth satellites orbiting near the earth's surface (about 20,000 mph), the mass correction factor (M/M_0) is still very nearly one (1.0000000005). Microscopic (or atomic) particles such as protons or electrons have been accelerated in laboratories to speeds approaching c, and in such work Eq. (2.9) has been well verified. Values of (M/M_0) of greater than 10^3 have been achieved. It is worth noting that the language used to describe the laws of mechanics at relativistic (nearly equal to c) speeds uses concepts that first become familiar in the relationships and equations of low-velocity mechanics. Our development will continue in the nonrelativistic domain and the conservation laws so exposed will be applicable to relativistic mechanics as well.

2.4.2 Density.

The different glide blocks in Fig. 2.10 have the same mass, but they vary greatly in size. It is clear then that size alone does not determine the mass of an object. The relationship between the mass of an object and its size, as measured by its volume, is expressed in terms of the mass density of that material. From daily life, we are aware of high-density and low-density materials. A block of wood is easier for a child to move about than a block of iron of the same size.

We know that if we made three aluminum glide blocks of identical size the three blocks would have the same mass. The amount of mass in a given volume of aluminum is a property of aluminum. Copper has a different characteristic mass in the same volume of copper.

The density of a material depends upon its composition, its temperature, and the pressure. For many common solids and liquids, the composition alone is sufficient to determine the density to within a few percent. If a small element of volume (Δ vol.) of material contains an amount of mass (Δm), the density of that material is defined as

$$\text{density} = \frac{(\Delta m)}{(\Delta\ \text{vol})}.$$

The usual units of density are kilograms per cubic meter, kg(m^{-3}), or grams per cubic centimeter, g(cm^{-3}). The units of g(cm^{-3}) are most often used for convenience. The density of water at 4°C is 1000 kg(m^{-3}) and the density of, for example, aluminum is 2700 kg(m^{-3}). In the cgs system, these densities are 1.000 g(cm^{-3}) and 2.700 g(cm^{-3}). The relative lengths of the glide blocks shown in Fig. 2.10 can be determined from the densities of the materials which are for gold, 19.3 g(cm^{-3}); copper, 8.92 g(cm^{-3}); aluminum, 2.70 g(cm^{-3}); and Lucite plastic, 1.18 g(cm^{-3}).

2.5 CONSERVATION OF MOMENTUM

Return again to the two-body explosion as produced on the air track. If the objects have masses M_A and M_B, and if the system starts from rest, then the objects A and B move apart with speeds $|\vec{v}_A|$ and $|\vec{v}_B|$, respectively. We have seen that

$$\frac{|\vec{v}_A|}{|\vec{v}_B|} = \frac{M_B}{M_A},$$

and

$$M_A|\vec{v}_A| = M_B|\vec{v}_B|. \qquad (2.10)$$

Again, as in Eq. (2.2), we recognize a left-right or A-B symmetry in the Eq. (2.10) above. The product $M|\vec{v}|$ of object A equals the product $M|\vec{v}|$ for object B. Note also that only one number, the mass, is needed to characterize each object in this experiment.

Remembering that $\vec{v}_A$ and $\vec{v}_B$ are vectors in opposite directions, Eq. (2.10) can be written to contain this information, or

$$M_A\vec{v}_A = -M_B\vec{v}_B,$$

and then

$$M_A\vec{v}_A + M_B\vec{v}_B = 0. \qquad (2.11)$$

Equation (2.11) is written as a conservation law, and the quantity that is conserved in this explosion is the sum of the products of $M\vec{v}$ of the individual objects. The product $M\vec{v}$ for a particle, the product of the mass of an object times its velocity, is called the momentum of that object.

Momentum is a vector quantity. Since mass is a scalar and velocity a vector, the product of the two is a vector. The direction of the momentum is the same as the direction of the velocity. The symbol $\vec{p}$ is often used for momentum:

$$\vec{p} = m\vec{v}. \qquad (2.12)$$

The units of momentum are (kg m/sec) in the mks system or (g cm/sec) in the cgs system.

Equation (2.11) states then that the total momentum of this system is conserved. This is our first encounter with this great law of physics, the law of conservation of momentum.

In words, the Eq. (2.11) states that the vector sum of the momenta of the individual objects remains constant following the two-body explosion. Each object had zero momentum before the event, so that the total momentum must be zero, and being conserved, remain constant at a value of zero.

The law of conservation of momentum states that

> For any isolated system, the total momentum of that system is constant.

The words "total momentum" mean the sum of all of the individual momenta of all the parts of the system. This sum must necessarily be a vector sum since the momentum of each part (or particle) of the system is a vector.

The total momentum of an isolated system remains constant as time develops independent of any kind of chemical, mechanical, electrical, or yet unnamed change that may occur within that system. The parts of the system may be flying apart from one another, and even though they become separated by very large distances, they still remain parts of the system. If the various parts of the system never do interact in any way with part or all of any other system, then the system under consideration remains isolated and the total momentum is constant.

An isolated system is one on which no external force acts. A complete discussion of what is meant by "isolated" will have to await further development of the concept of force.

When Eq. (2.11) is written in the form given,

$$M_A\vec{v}_A + M_B\vec{v}_B = 0,$$

the equation is expressed as a conservation law. The total momentum of this system is zero, and remains zero so long as the system is isolated from the rest of the universe. The fact that the sum of the momenta of the particles of this system must be zero, and not some other number, is a consequence of the special experiment under consideration. If the system (blocks A and B) were not at rest, but the blocks were moving together before the explosion with the spring compressed between them, then the total momentum would not be zero. The total momentum would have to equal the sum of the initial momenta of the system. We consider this next in section 2.6, where we examine a moving explosion.

We will see in section 2.6 that the fact that momentum is conserved at zero in one reference frame means that momentum is conserved at some value other than zero in any frame moving at constant velocity relative to the frame in which the total momentum is zero. The total momentum is conserved (remains constant), in each frame, in accord with the principle of relativity.

2.6 THE PRINCIPLE OF RELATIVITY AND THE CONSERVATION LAW

2.6.1 The Moving Explosion.

In Chapter 1 we discussed the principle of relativity, which states that the laws of nature must be the same in two different laboratories or frames of reference that move relative to one another with a constant velocity.

If the conservation of momentum law holds for an explosion in our frame, then it must also hold for explosions in other frames moving at constant velocity relative to ours. According to the relativity principle, no matter in which of these frames observers make measurements on the air-track explosion, all observers would conclude that the total momentum of the system is conserved.

It may be helpful to imagine that the air track is fixed on a large cart which always moves horizontally with constant speed in the +x direction. The x axis is fixed to the laboratory floor. There is room on the cart for air-track explosion equipment including timers and students for making

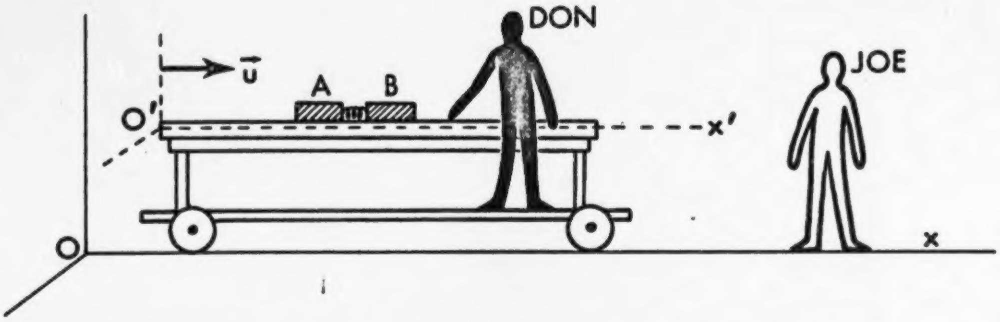


Fig. 2.11 The two-particle explosion under investigation by two sets of observers symbolized by Joe and Don. Don, who is on the cart, is at rest with respect to the air track. The cart velocity is $\vec{u}$ relative to Joe, who is on the floor. The coordinate axes x' and x are parallel.

measurements. The student observers on the cart locate block positions on the x' axis which is located on the air track, and the x' axis is parallel to the x axis. The cart observers measure velocities in terms of time rate of change of the x' coordinate, and the symbol $\vec{v}$ will be used for their velocities.

For quick reference, call the set of observers on the cart (x') by the name of one observer, Don, and the set on floor (x axis) by the name of one, Joe. The cart moves with velocity u which is directed in the +x direction. The motion is indicated by the vector $\vec{u}$, shown in Fig. 2.11.

Don studies explosions, and observes that two given blocks of mass M_A and M_B, acquire velocities $\vec{v}_A'$ and $\vec{v}_B'$ in opposite directions relative to the track (x' axis). In Don's reference frame, the blocks are at rest before the explosion. Don's situation is just exactly the experimental setup described in section 2.4. Don's observations will be the same as those of section 2.4 and his measurements show

$$\frac{|\vec{v}_A'|}{|\vec{v}_B'|} = \frac{M_B}{M_A}.$$

These results can be expressed in vector form, as in section 2.5,

$$M_A\vec{v}_A' + M_B\vec{v}_B' = 0. \qquad (2.13)$$

Conservation of total momentum of the A-B system in Don's frame is expressed by Eq. (2.13).

How would Joe, who has his own meter sticks and clocks, describe the explosion? Before the spring between blocks A and B is released, the blocks both have the same velocity, $\vec{u}$, which is just the velocity of the cart (or x' axis) since both blocks are at rest in that frame. Immediately after the completion of the explosion, A and B have different velocities relative to Joe. The velocities measured by Joe are $\vec{v}_A$ and $\vec{v}_B$.

If a block moves with velocity $\vec{v}'$ along the x' axis, and the x' axis moves with respect to the x axis with velocity $\vec{u}$, then the velocity of that block relative to the floor is, from Eq. (1.11),

$$\vec{v} = \vec{v}' + \vec{u}. \qquad (2.14)$$

For example, if the cart velocity $\vec{u}$ = +3m/sec (3m/sec in the positive x direction) and if B moves with velocity $\vec{v}_B'$ = +1.5m/sec relative to the cart, then $\vec{v}_B$ = +3m/sec + 1.5m/sec = +4.5 m/sec relative to the x axis. Remember that the symbols in Eq. (2.13) carry their own sign which signifies the direction of the vector along the x' axis. If one of the velocities is negative it appears as a negative number.

From Eq. (2.13),

$$\vec{v}' = \vec{v} - \vec{u}. \qquad (2.14a)$$

Write Eq. (2.14a) separately for block A and B.

$$\vec{v}_A' = \vec{v}_A - \vec{u}.$$
$$\vec{v}_B' = \vec{v}_B - \vec{u}. \qquad (2.15)$$

Equations (2.15) make the connection between what measurements Don makes ($\vec{v}_A'$, $\vec{v}_B'$) and the measurements Joe makes ($\vec{v}_A$, $\vec{v}_B$). This connection or transformation is an expression of the fact that Don and Joe agree on space and time measurements. It is the Galilean transformation developed in section 1.5.

Don's velocity data fit Eq. (2.13). If we substitute the right side of Eq. (2.15) into Eq. (2.13) for each $\vec{v}'$, then Eq. (2.13) becomes

$$M_A(\vec{v}_A - \vec{u}) + M_B(\vec{v}_B - \vec{u}) = 0$$

or

$$M_A\vec{v}_A + M_B\vec{v}_B = M_A\vec{u} + M_B\vec{u}. \qquad (2.16)$$

Equation (2.16) is our prediction of the equation that Joe's measurements should obey.

Can you recognize the conservation of momentum law being expressed by Eq. (2.16)? The right side of Eq. (2.16) is the vector sum of the momenta of A and B before the explosion and the left side of Eq. (2.16) is the vector sum of the momenta of A and B after the explosion. The total momentum remains constant, or momentum is conserved in Joe's reference frame as well as Don's.[8]

Example: Put numbers into the above.

Given: $M_A = 0.3 \text{ kg}$ $\quad M_B = 0.2 \text{ kg}$

$\vec{v}_A' = -1.0 \text{ m/sec}$ $\quad \vec{u} = +3 \text{ m/sec}$

(1) Find $\vec{v}_B'$.

$$M_A\vec{v}_A' + M_B\vec{v}_B' = 0$$

$$(.3 \text{ kg})(-1.0 \text{ m/sec}) + (0.2 \text{ kg})\, \vec{v}_B' = 0$$

$$\boxed{v_B' = +1.5 \text{ m/sec}}$$

(2) Find $\vec{v}_A$, $\vec{v}_B$

for A, $\quad \vec{v}_A' = \vec{v}_A - \vec{u}$

$$-1.0 \text{ m/sec} = \vec{v}_A - 3 \text{ m/sec}$$

$$\boxed{\therefore \ \vec{v}_A = +2 \text{ m/sec}}$$

$$\vec{v}_B' = \vec{v}_B - \vec{u}$$

$$1.5 \text{ m/sec} = \vec{v}_B - 3 \text{ m/sec}$$

$$\boxed{\therefore \ \vec{v}_B = +4.5 \text{ m/sec}}$$

(3) What is the momentum in the unprime (Joe's) frame before the explosion?

$$\vec{P}_{TOT} = M_A\vec{v}_{A(\text{before})} + M_B\vec{v}_{B(\text{before})}$$

We know that

$$\vec{v}_{A(\text{before})} = \vec{v}_{B(\text{before})} = \vec{u}.$$

$$\begin{aligned} \therefore \ \vec{P}_{TOT} &= M_A\vec{u} + M_B\vec{u} \\ &= (.3 \text{ kg})(3\text{m/sec}) \\ &\quad + (.2 \text{ kg})(3\text{m/sec}) \\ \vec{P}_{TOT} &= +\ 1.5 \text{ kg m/sec} \end{aligned}$$

(4) What is the momentum in the unprime frame after the explosion?

$$\begin{aligned} \vec{P}_{TOT} &= M_A\vec{v}_A + M_B\vec{v}_B \\ &= (.3 \text{ kg})(2 \text{ m/sec}) \\ &\quad + (0.2 \text{ kg})(4.5 \text{ m/sec}) \\ \vec{P}_{TOT} &= +1.5 \text{ kg m/sec} \end{aligned}$$

Note that the total (vector sum) momentum of the system remains constant. Momentum is conserved.

How would Joe describe this explosion? He would say that the two blocks were initially moving together with the same speed of +3 m/sec. The spring released and after the "explosion," A was still moving to the right (+x direction), but more slowly at +2 m/sec and B was moving more rapidly at +4.5 m/sec.

Don says the two blocks were initially at rest and after the spring released A moved left (−x' direction) at −1 m/sec and B moved right (+ x' direction) at +1.5 m/sec.

Both sets of observers' measurements are in accord with the law of conservation of momentum.

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[8] The physics of this thought experiment is contained in the two-particle explosion in which both particles are moving together before the explosion. The "system" consists of particles A and B in addition to a small spring the motion of which is assumed to be negligible. Remember that the air track is not part of the system. Since the system is well isolated from the air track, it is really not necessary to have the air track itself in the moving (Don's) frame. We could just as well do the experiment in which the air track was fixed in the x frame (Joe's) and in which Don makes observations from a cart moving with velocity $\vec{u}$ equal to the common initial velocity of A and B. The results in the form of Eqs. (2.13) and (2.16) would still obtain.

Is it possible for Joe, with only a knowledge of his own measurements on the explosion and a knowledge of Galilean relativity, to predict what Don's measurements are?

2.6.2 The Sticky Collision.

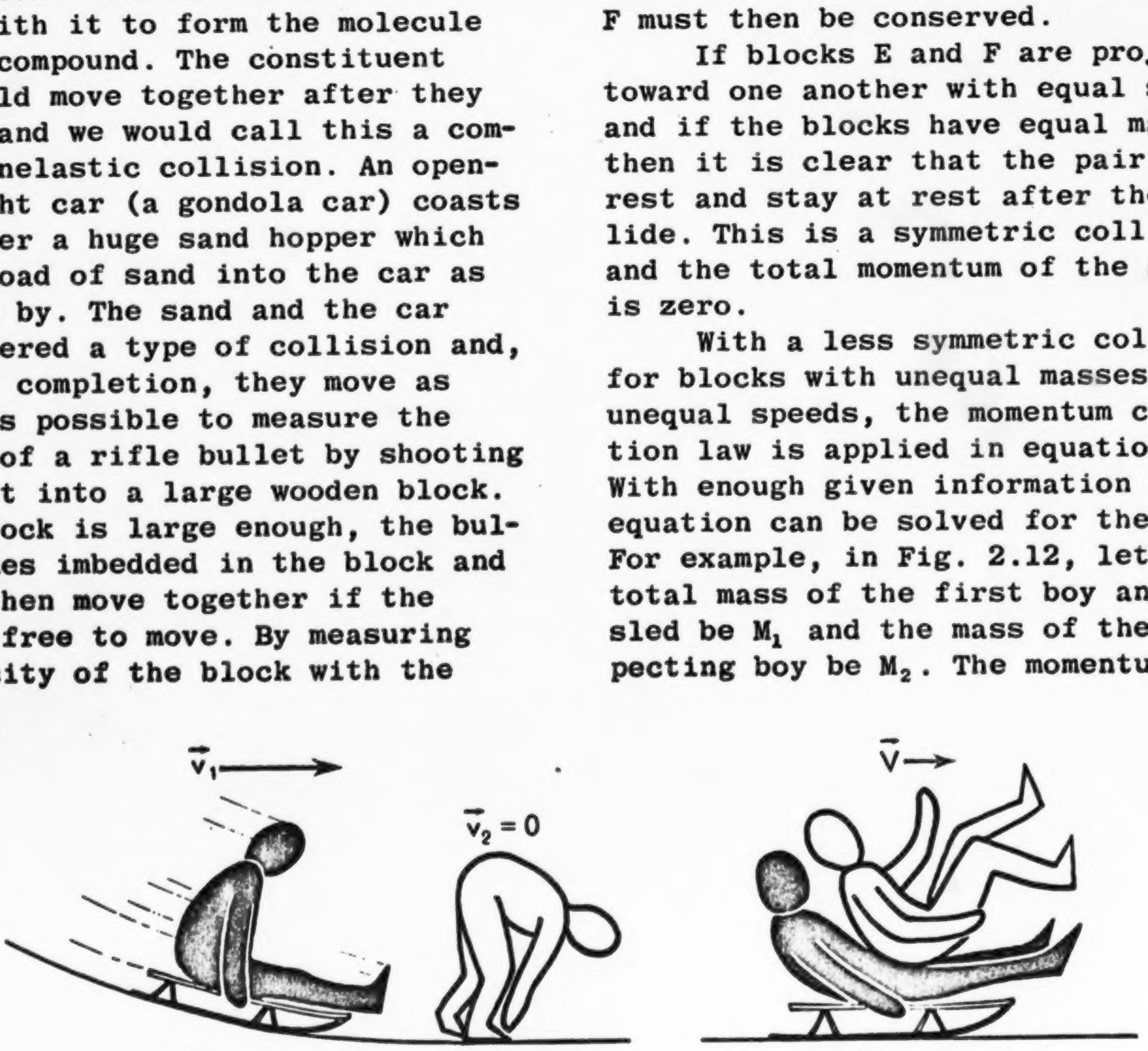
Let us consider a problem that is something like an inverted explosion. Assume that the ends of the sliding blocks have been prepared in such a way that the blocks stick together whenever the ends touch one another. The ends might be coated with a glue or mastic, or some kind of mechanical coupling might be used. When two such blocks are projected toward one another, they collide, join together, and move as one. Such sticky collisions are usually referred to as completely inelastic collisions.

Perhaps you can think of some examples of completely inelastic collisions. In chemical reactions, one molecule or atom might come in contact with a second molecule or atom and combine with it to form the molecule of a new compound. The constituent parts would move together after they combine, and we would call this a completely inelastic collision. An open-top freight car (a gondola car) coasts along under a huge sand hopper which drops a load of sand into the car as it coasts by. The sand and the car have suffered a type of collision and, after its completion, they move as one. It is possible to measure the velocity of a rifle bullet by shooting the bullet into a large wooden block. If the block is large enough, the bullet becomes imbedded in the block and the two then move together if the block is free to move. By measuring the velocity of the block with the bullet imbedded in it, and by knowing the mass of the block and the mass of the bullet, the original speed of the bullet can be found from the law of conservation of momentum applied to this completely inelastic collision. Figure 2.12 shows another example of a completely inelastic collision.

Consider blocks E and F, of mass M_E and M_F, respectively, which are projected toward one another on our air track. Assume that these blocks suffer a completely inelastic collision. It will not be necessary to understand the nature of the glue used as a sticking agent or how the mechanical coupling works. We need only establish that our two-block system is isolated in that the blocks move frictionlessly over the track. The total momentum of the system of blocks E and F must then be conserved.

If blocks E and F are projected toward one another with equal speeds, and if the blocks have equal mass, then it is clear that the pair come to rest and stay at rest after they collide. This is a symmetric collision and the total momentum of the system is zero.

With a less symmetric collision for blocks with unequal masses and/or unequal speeds, the momentum conservation law is applied in equation form. With enough given information the equation can be solved for the unknown. For example, in Fig. 2.12, let the total mass of the first boy and his sled be M_1 and the mass of the unsuspecting boy be M_2. The momentum of



(a) (b)

Fig. 2.12 (a) A completely inelastic collision in which one object (the unsuspecting boy) is initially at rest. (b) After the collision the system moves with one velocity $\vec{V}$.

this system before the collision is $M_1\vec{v}_1$ where $\vec{v}_1$ is the initial velocity of the sled. The total momentum after the collision is $(M_1 + M_2)\vec{V}$. In applying the conservation of momentum law, we write

$$M_1\vec{v}_1 = (M_1 + M_2)\vec{V}.$$

If M_1, M_2, and $\vec{v}_1$ are given, then the final velocity is found to be

$$\vec{V} = \frac{M_1}{(M_1 + M_2)}\vec{v}_1 .$$

In the completely inelastic collision on the air track, assume that masses M_E and M_F have initial velocities $\vec{v}_E$ and $\vec{v}_F$. After the collision M_E and M_F move together with velocity $\vec{V}$.

The total momentum before the collision is

$$\vec{P}_{TOT} = M_E\vec{v}_E + M_F\vec{v}_F,$$

and after the collision

$$\vec{P}_{TOT} = (M_E + M_F)\vec{V}.$$

Momentum of this isolated system is conserved and the conservation is expressed in Eq. (2.17):

$$M_E\vec{v}_E + M_F\vec{v}_F = (M_E + M_F)\vec{V}. \quad (2.17)$$

Numerical Example:

In a one-dimensional collision along the x axis, let

$$M_E = 0.25 \text{ kg}; \; \vec{v}_E = +2 \text{ m/sec},$$

and

$$M_F = 0.75 \text{ kg}; \; \vec{v}_F = -4 \text{ m/sec}.$$

Find V, the final velocity.

$$M_E\vec{v}_E + M_F\vec{v}_F = (M_E + M_F)\vec{V}$$

$$(0.25 \text{ kg})(2 \text{ m/sec}) + (0.75 \text{ kg})(-4\text{m/sec}) = (1.00 \text{ kg})V$$

$$\vec{V} = -2.5 \text{ m/sec}.$$

After the collision, the two move with a speed of 2 m/sec in the negative x direction. M_F had more negative momentum than M_E had positive momentum, and the total momentum in this x reference frame is negative.

It is interesting to note that there is an inertial reference frame moving with constant velocity relative to the laboratory in which the two blocks are at rest after the collision. In this frame the final momentum is therefore zero. It will be shown by explicit calculation that the momentum is conserved in this frame as well.

Call this frame the primed frame and put its x' axis parallel to the laboratory x axis along the air track.

Let the constant velocity of the x' frame with respect to the x frame be $\vec{u}$ and since the blocks move with velocity $\vec{V}$ after the collision,

$$\vec{u} = \vec{V}.$$

Relative to the primed frame, M_E and M_F move before the collision with velocities $\vec{v}_E{}'$ and $\vec{v}_F{}'$ which are related to $\vec{v}_E$ and $\vec{v}_F$ by the transformation rule from Eq. (1.11).

$$\vec{v}_E{}' = \vec{v}_E - \vec{u}$$

and $\quad (2.18)$

$$\vec{v}_F{}' = \vec{v}_F - \vec{u}.$$

In the primed frame the total momentum of the system before the collision is $\vec{P}'_{TOT}$ where,

$$\begin{aligned}\vec{P}'_{TOT} &= M_E\vec{v}_E{}' + M_F\vec{v}_F{}' \\ &= M_E(\vec{v}_E - \vec{u}) + M_F(\vec{v}_F - \vec{u})\end{aligned}$$

$$\vec{P}'_{TOT} = M_E\vec{v}_E + M_F\vec{v}_F - (M_E + M_F)\vec{u}. \quad (2.19)$$

Since $\vec{u} = \vec{V}$, Eq. (2.19) becomes

$$\vec{P}'_{TOT} = M_E\vec{v}_E + M_F\vec{v}_F - (M_E + M_F)\vec{V}. \quad (2.20)$$

From Eq. (2.17) it follows that

the right-hand side of Eq. (2.20) is zero.

$$\therefore \quad \vec{P}'_{TOT} = 0. \qquad (2.21)$$

Equation (2.21) expresses the total momentum of the E,F system before the collision in the x' reference frame. The Eq. (2.21) was calculated from the knowledge (1) that momentum is conserved in the unprimed x frame, and (2) that the velocities of E and F in the primed frame could be expressed in terms of their velocities in the unprimed frame in Eq. (2.18). The result of the calculation is that the total momentum of the system in the primed frame x' before the collision is found to be zero, Eq. (2.21). We know that $\vec{P}'_{TOT}$ is zero after the collision since the blocks are then at rest in the primed frame. This shows that the momentum is conserved in the primed frame as well as in the unprimed frame.

Let us rework the numerical example for blocks E and F in the primed frame. The x' axis moves relative to the x axis with a velocity

$$\vec{u} = \vec{V} = -2.5 \text{ m/sec}.$$

Then

$$\vec{v}_E{}' = \vec{v}_E - \vec{u}$$
$$\vec{v}_E{}' = +2.0 \text{ m/sec} - (-2.5 \text{ m/sec})$$
$$\vec{v}_E{}' = +4.5 \text{ m/sec},$$

and

$$\vec{v}_F{}' = \vec{v}_F - \vec{u}$$
$$\vec{v}_F{}' = -4 \text{ m/sec} - (-2.5 \text{ m/sec})$$
$$\vec{v}_F{}' = -1.5 \text{ m/sec}.$$

In this reference, frame E is initially moving in the +x' direction at 4.5 m/sec and F is moving in the −x' direction at 1.5 m/sec.

This particular frame of reference is of more than usual interest, for this frame contains the center of mass of the system.

2.7 THE CENTER-OF-MASS CONCEPT

In Chapter 1 the law of inertia was stated for complicated systems as a law describing the motion of the center-of-mass point. In an isolated system, it is this point which moves with constant velocity during the complicated motions of the system's parts. In the last section it was stated that in the inertial reference frame that contains the center of mass, the total momentum of the system is zero. Even though this is one of many inertial frames, the fact that zero is a rather unique number leads one to suspect that this reference frame is particularly simple for the description of motion, and it is

The center-of-mass concept is not a completely unfamiliar one. We use it commonly in what seem to be "natural" descriptions of the internal motions or motions within a system. A few examples will recall the common distinction between motion of an object as a whole and the internal motions of an object. When we see a bird flying we say that the bird "flaps his wings," but that is looking at the wing motion from the bird's viewpoint. The wing-flapping motion is an internal motion, a motion within the (bird) system. As seen from the ground, the motion of a wing describes a wavy line as shown in Fig. 2.13.

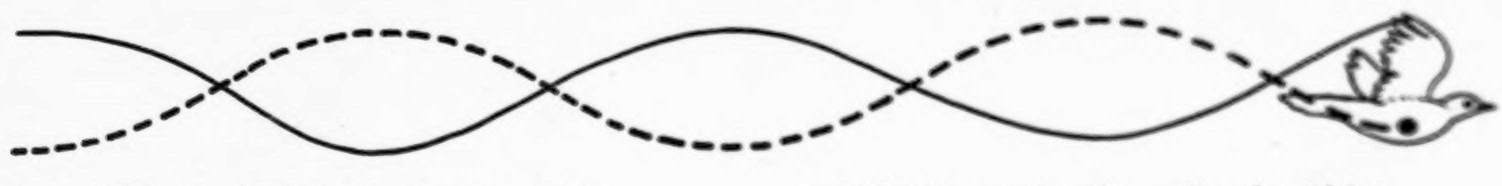


Fig. 2.13 An idealized bird in flight. The dotted line marks the motion of the center of mass point of the bird and the solid line marks the path of a white feather at the bird's wing tip.

When we think of a flapping wing we think of up-down motion of the wing, and the up-down motion is the wing motion in the reference frame of the bird. It is the motion in the center of mass reference frame.

As a helicopter ascends, we think of the whole helicopter rising with the great whirling propeller spinning above it. We tend to think of the propeller blades spinning relative to the body of the helicopter. Actually the path of a point on the tip of that great propeller in the air is a helix, or spiral, as shown in Fig. 2.18.

In the study of mechanics and in the study of the conservation laws it is often useful and illuminating to separate internal motion from motion of the object as a whole. The most useful point of reference of a system about which to describe the motion within that system is the center-of-mass point.

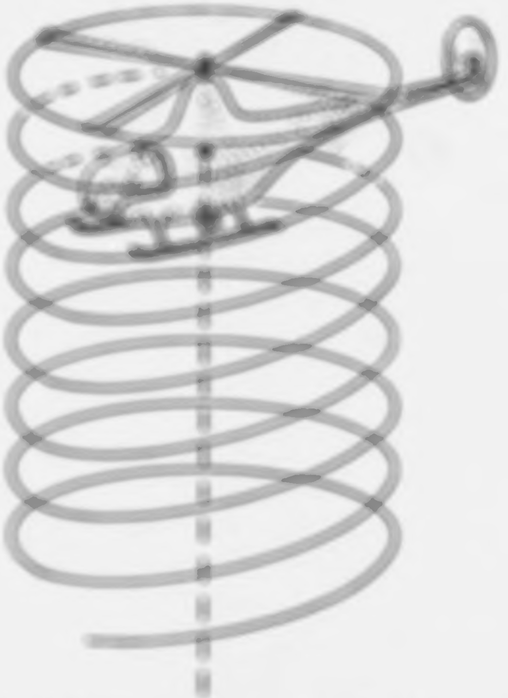


Fig. 2.18 A helicopter in vertical take-off. The dotted line marks the path of the center of mass of the helicopter and the solid line marks the path of one tip of one of the propeller blades.

2.7.1 The Center-of-Mass Point.

We would expect that this point would move along with the system and be somewhere in the "middle" of it. One might first think that the point would be located at the average of positions of all of the parts of the system. This thought is partly right. In averaging the positions of each part of the system, the parts with more mass are counted more. For this kind of averaging, the average position is called the center-of-mass point. It may not be at the geometric center of the system. If some parts have greater mass, the center-of-mass will be shifted toward the massive parts away from the geometric center.

The formal definition of the location of the center-of-mass point can be specified in relation to any convenient coordinate system. The position of the center of mass is the weighted average of the positions of all the elements of mass composing that system, with the mass of each element used as a weighting factor.

Let $\vec{r}_c$ be the position of the center of mass of a system or object, and let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_i, \ldots \vec{r}_N$ be the position vectors of each of the N elements of mass which comprise the system. Let $m_1, m_2, m_3, \ldots, m_i, \ldots m_N$ be the masses of the respective mass elements.

Then by the definition,

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_i\vec{r}_i + \ldots + m_N\vec{r}_N}{m_1 + m_2 + \ldots + m_i + \ldots + m_N}$$

or

$$\vec{r}_c = \frac{\sum\limits_{(i=1)}^{N} m_i\vec{r}_i}{\sum\limits_{(i=1)}^{N} m_i}, \qquad (2.20)$$

where N is the total number of mass elements in the system.

(The symbol, Σ, is shorthand for "the sum of." Each term of the summation is represented by the subscript "i," where "i" is a running index covering a range of values

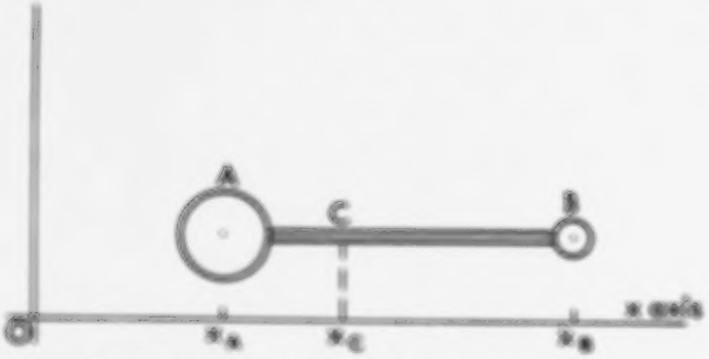


Fig. 2.15 Two metal spheres A and B connected by a thin plastic rod. The x axis is parallel to the rod.

from one (1) through N. The range of values of "i" over which the sum is made is indicated by the subscript "i=1" below Σ and the superscript "N" above Σ.)

The denominator of Eq. (2.22) equals the total mass of the system. Equation (2.22) is then

$$M\bar{r}_c = \sum_{i=1}^{N} m_i \bar{r}_i, \qquad (2.23)$$

where

$$M = \sum_{i=1}^{N} m_i.$$

Before studying some examples of the application of this definition, it will be useful to have Eq. (2.23) expressed in terms of individual x, y, and z coordinates of a Cartesian coordinate system. These are

$$x_c = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

$$y_c = \frac{\sum_{i=1}^{N} m_i y_i}{M} \qquad (2.24)$$

$$z_c = \frac{\sum_{i=1}^{N} m_i z_i}{M}.$$

As an introductory specific example, in obtaining the center-of-mass position, think of an object shaped like a dumbbell made of two homogeneous metal spheres, A and B, having masses, M_A and M_B, respectively, connected by a thin plastic rod. Assume that the mass of the plastic rod can be neglected for the purposes of this discussion. The system is shown in Fig. 2.15.

There are two mass elements in this system, M_A and M_B, and their positions are identified as the positions of the centers of the spheres. (This means that M_A and M_B are assumed to be point masses located at the center of the spheres. This assumption is justified later in this section.)

If $M_A > M_B$, then the center-of-mass point will be located closer to A than to B. Let the center-to-center distance between A and B be represented by L. This is a one-dimensional system so that, for convenience, one coordinate axis (x) is established parallel to the rod connecting A and B. Then

$$x_B - x_A = L,$$

and the center-of-mass coordinate x_C is

$$x_C = \frac{M_A x_A + M_B x_B}{(M_A + M_B)}.$$

Suppose $M_A = 2M_B$; then

$$x_C = \frac{2M_B x_A + M_B x_B}{(2M_B + M_B)}$$

$$x_C = \frac{2}{3}x_A + \frac{1}{3}x_B.$$

Since

$$x_B = x_A + L,$$

then

$$x_C = x_A + L/3.$$

The center-of-mass point, C, is located between A and B and one third of the spacing, L, away from A. The ratio of the distances from C to B and

A is two to one and this, clearly, reflects the mass ratio.

$$\frac{x_B - x_C}{x_C - x_A} = \frac{M_A}{M_B} = \frac{2}{1}.$$

In general, one can find a coordinate system that will have its origin at the center of mass even though the location of that point may not yet be known. This is done by requiring that $\bar{r}_C$ equal zero in this coordinate system. In the example just considered, let x' be the coordinate axis which has its origin at C.

Then

$$x_C' = \frac{M_A x_A' + M_B x_B'}{(M_A + M_B)}. \qquad (2.25)$$

Now, since we have set $x_C' = 0$, and if we choose the system shown in Fig. 2.16 with $M_A = 2M_B$, Eq. (2.25) becomes

$$0 = 2x_A' + x_B'. \qquad (2.26)$$

The origin of the x' axis must be located in such a way that Eq. (2.26) is obeyed. Also, since the spacing between B and A is equal to L,

$$x_B' - x_A' = L. \qquad (2.27)$$

It follows from Eqs. (2.26) and (2.27) that,

$$x_A' = -L/3$$

$$x_B' = +2L/3.$$

The x' axis with points x_A' and x_B' marked is shown in Fig. 2.16. The thin plastic rod connecting A and B in our example is actually unnecessary, for if it were removed the center-of-mass point would still be located at C. It is not necessary that there actually be some material at the center-of-mass point. The position is clearly specified without a marker being there.

The location of the center of mass of more complicated objects is determined by application of Eq. (2.23). If the system has some element or elements of symmetry, the symmetry can be used to help locate the center of mass. If the system has a plane of symmetry, the center of mass will lie in the plane.

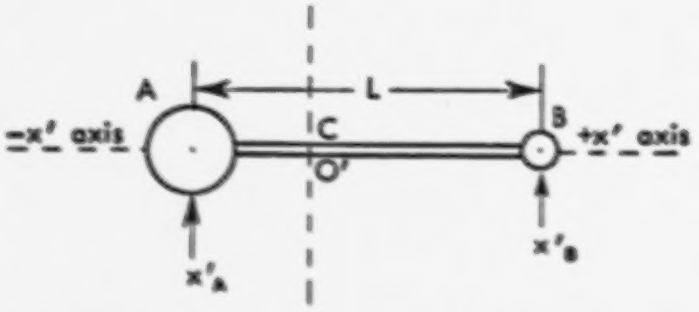


Fig. 2.16 The origin (O') of the x' axis is established at C. With $M_A = 2M_B$, $x_B' = +2L/3$ and $x_A' = -L/3$ where L is the center-to-center spacing of A and B.

For example, a tennis racket is symmetric about two planes that are parallel to the handle and perpendicular to one another. Since the center of mass must be on both of these planes, it is on the line of intersection of these planes. The location is shown in Fig. 2.17.

A frying pan has only one plane of symmetry. Can you describe it?

The symmetry of the water molecule can be used to good advantage in locating its center-of-mass point. The water molecule is approximated as three points in Fig. 2.18. Its center of mass lies along the bisector of the 105° angle shown as a dotted line in Fig. 2.18.

Prove to yourself that the center of mass is located on the bisector of the angle HOH in Fig. 2.18 and at a distance OC = (R/9) cos 52.5° where R is the O – H spacing in the molecule.

Sometimes there are lumps or groupings of matter in the system that are more or less distinguishable from one another. All of the mass elements of such a group can be represented by a point mass, equal to the group mass, located at the center-of-mass point in the group. We did this in locating the center of mass of the dumbbell-shaped object in an earlier

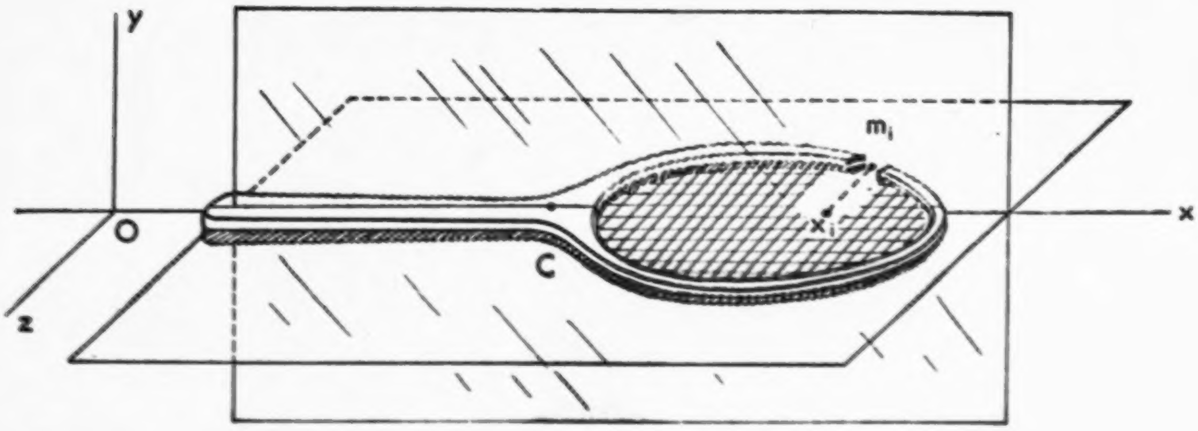


Fig. 2.17 The xy and xz planes are the mutually perpendicular planes of symmetry of a tennis racket. The line of intersection containing C, the center of mass point, is taken as the x axis. A typical mass element, m_i, is shown. The use of the symmetry has established the y and z coordinates of C and only the x component is left to be determined.

discussion. Each sphere, M_A and M_B, was represented as a point mass located at the center of each sphere. This trick is especially useful when the composite parts of the system have easily determined centers of mass, but the whole system may not be very symmetric.

Suppose we want to find the center of mass of the object shown in Fig. 2.19, which is a first approximation to a hockey stick.

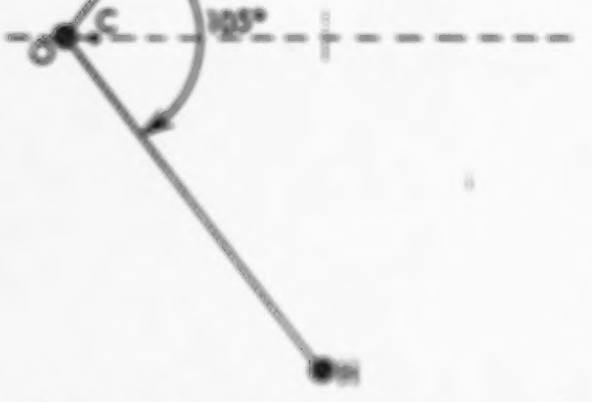


Fig. 2.18 Schematic representation of a water molecule. Point C is the center-of-mass point. The mass of the oxygen atom, O, is 16 times the mass of one of the hydrogen atoms, H. The OH distance is R, and OC = (1/9)R cos 52.5°.

The centers of mass of the handle A and the blade B are at their respective geometric centers. To find the center of mass of the hockey stick, replace the handle with a point mass M_A located at its center of mass C_A, and replace the blade with a point mass M_B located at its center of mass C_B. The center of mass of the whole hockey stick is then located by finding the center of mass of the M_A and M_B point masses.

This procedure is exact and the formal definition of center of mass

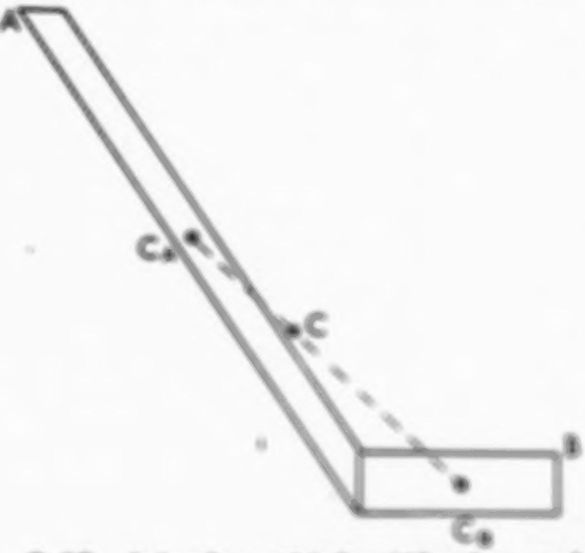


Fig. 2.19 A hockey stick without rounded edges and corners. The center of mass of part A is located at C_A and the center of mass of part B is located at C_B. The center of mass of the hockey stick is located at point C on the line connecting C_A and C_B.

can be used to demonstrate its validity. The center-of-mass position is defined by Eq. (2.23),

$$M\vec{r}_c = \sum_{i=1}^{N} m_i\vec{r}_i .$$

Divide the summation into two parts involving in one only mass elements of A and in the other only mass elements of B. Then Eq. (2.23) can be written as

$$M\vec{r}_c = \sum_{A} m_i\vec{r}_i + \sum_{B} m_i\vec{r}_i ,$$

where the A and B mean "over part A only" and "over part B only" respectively. The term $\sum_{A} m_i\vec{r}_i$ is just the summation we would have if we were finding the center of mass of part A only. Or, we can write

$$M_A\vec{r}_A = \sum_{A} m_i\vec{r}_i \qquad (2.28)$$

and

$$M_B\vec{r}_B = \sum_{B} m_i\vec{r}_i, \qquad (2.29)$$

where $\vec{r}_A$ and $\vec{r}_B$ are position vectors of the center-of-mass points of A and B, respectively.

Each individual summation can be replaced by the product of a point mass and the position vector of its center of mass.

Finally,

$$M\vec{r}_c = M_A\vec{r}_A + M_B\vec{r}_B .$$

The argument is readily extended to systems with more than two parts.

2.7.2 *Center of Mass and the Motion of the System.*

A useful property of the motion of complex systems is that the total momentum of the system equals the product of the total mass of the system and the velocity of the center-of-mass point. A corollary to this property is that the internal motions of the system do not contribute to the total linear momentum of the system. These characteristics can be understood from a description of the center-of-mass motion of the system. Even though some or all of the elements of mass in the system may be moving, the instantaneous position of the center-of-mass point can be established from the instantaneous locations of the mass elements. Figure 2.20 gives an excellent example of this.

Fig. 2.20 is a stroboscopic ("time history"), photograph of the motion of a dumbbell-shaped object similar to the one discussed above. One mass, indicated by an open circle (o) has twice the mass of the other, shown as a cross (x) in the photograph. The center of mass is located at a distance equal to one third of the two mass center-to-center spacing away from the larger mass. A closed circle indicates the center-of-mass position.

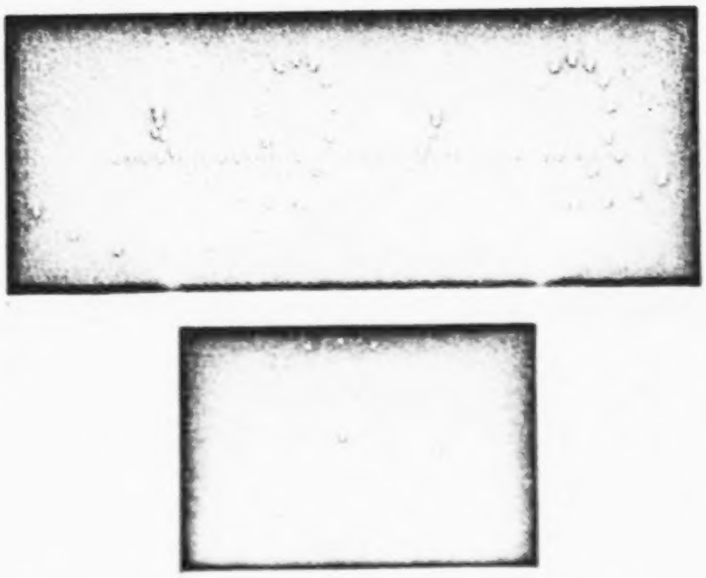


Fig. 2.20 In (a) of this figure, the photograph of subsequent positions of the dumbbell is made from a fixed camera while the dumbbell moves past it, rotating "end over end" as it goes. Each set of position indicators on the picture provides a simultaneous location of both masses and the center of mass.
In (b), the camera moves in the reference frame of the center of mass while the dumbbell repeats the motion shown in (a).
(Photos courtesy Film Studio, Educational Services, Incorporated)

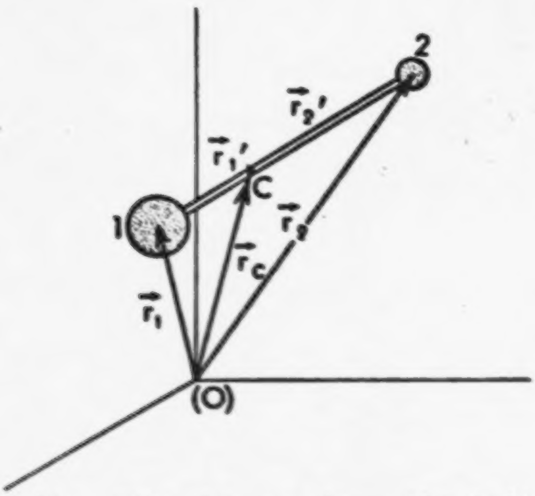


Fig. 2.21a This figure shows the initial instantaneous position of the dumbbell-shaped object in the (0) reference frame. Mass M_1 equals $2M_2$. The center-of-mass point is labeled C.

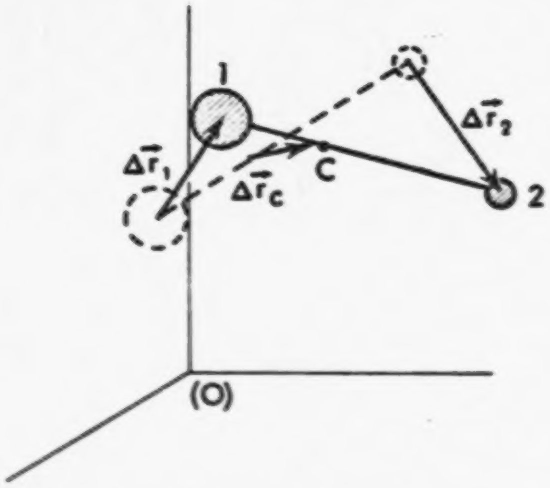


Fig. 2.21b Instantaneous position of dumbbell shown at a time Δt later than Fig. 2.21a. The displacements of M_1, M_2, and point C in the (0) reference frame are indicated by vectors $\Delta\bar{r}_1$, $\Delta\bar{r}_2$, and $\Delta\bar{r}_c$, respectively.

The motion in Fig. 2.20a looks a bit complicated, but the successive positions of the dumbbell-shaped object can be located easily. For every instantaneous position of the center of mass, the simultaneous positions of M_1 and M_2 can be located on a straight line passing through the center of mass. For each subsequent position of the center of mass, the angular position of this line is changed. It appears from Fig. 2.20a that the object is rotating in some way as it moves along. The rotational motion is more apparent in Fig. 2.20b.

The first thing that strikes one's eye in Fig. 2.20b is the circular motion of the x and o symbols. Clearly, the dumbbell is simply rotating at constant angular velocity in this frame of reference. To make the photograph of Fig. 2.20b, the experimenter had to move his camera at constant velocity in order to "stop" the center-of-mass point. The velocity needed for his camera could be obtained from Fig. 2.20a in which this velocity could be determined from the equal spacing of the center-of-mass points.

We conclude then that the motion in the figure is one of rotation about the center-of-mass point superimposed upon translation of the center of mass.

There is an appealing simplicity in the separation of motion into translation and rotation for this system. The rotational motion is the internal motion of this object. The point at which the division between the two motions is made is the center-of-mass point in the system. If we look again at the defining equation for the center-of-mass position $\bar{r}_c$ and see how $\bar{r}_c$ varies with time, we can see how this separation is conveniently made.

Let $\bar{r}_c$ be the instantaneous position vector to the center-of-mass point C, and let $\bar{r}_1$ and $\bar{r}_2$ be the simultaneous position vectors to masses (1) and (2), respectively, of the dumbbell used in Fig. 2.21. The location of an origin, (0), used in specifying these vectors, as shown in Fig. 2.21 is in an inertial frame of reference. The position of each mass element relative to the center of mass C is given by vectors in the center-of-mass coordinate system (0') having its origin at point C. These vectors are labeled $\bar{r}_1'$ and $\bar{r}_2'$ for masses (1) and (2) respectively.

For each mass element

$$\bar{r}_1 = \bar{r}_c + \bar{r}_1', \qquad (2.30$$

and

$$\bar{r}_2 = \bar{r}_c + \bar{r}_2'. \qquad (2.31$$

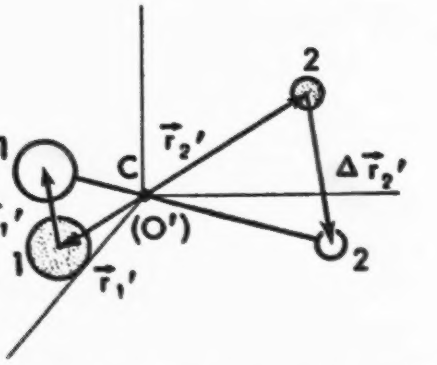


Fig. 2.21c Two instantaneous positions of the dumbbell are shown in the (O') reference frame. The origin of the (O') frame is located on the center-of-mass point C. The "internal displacements" measured relative to (O') are indicated by $\Delta\vec{r}_1'$ and $\Delta\vec{r}_2'$. Note that $\Delta\vec{r}_1'$ and $\Delta\vec{r}_2'$ are antiparallel in this two-mass system. What is the ratio of the magnitude of $\Delta\vec{r}_1'$ to the magnitude to $\Delta\vec{r}_2'$?

Multiplying Eq. (2.30) and Eq. (2.31) by M_1 and M_2, respectively, these equations become

$$M_1\vec{r}_1 = M_1\vec{r}_C + M_1\vec{r}_1' \qquad (2.32)$$

and

$$M_2\vec{r}_2 = M_2\vec{r}_C + M_2\vec{r}_2'. \qquad (2.33)$$

The sum of Eqs. (2.32) and (2.33) is

$$M_1\vec{r}_1 + M_2\vec{r}_2 = (M_1 + M_2)\vec{r}_C + [M_1\vec{r}_1' + M_2\vec{r}_2'], \qquad (2.34)$$

and from this sum we will be able to find the total momentum, $(M_1\vec{v}_1 + M_2\vec{v}_2)$.

Before doing so, it should be noted that the sum of terms in the bracket [ ] of Eq. (2.34) is zero. This must be true, because the bracket sum is the weighted (by mass) sum of the positions of the elements of mass composing this system where these positions (the primed vectors) are specified relative to C. From the definition of center of mass, it must be true that

$$M_1\vec{r}_1' + M_2\vec{r}_2' = (M_1 + M_2)\vec{r}_C'.$$

Since the origin of the (O') coordinate system is located at C, the center of mass,

$$\vec{r}_C' = 0.$$

$$\therefore\ [M_1\vec{r}_1' + M_2\vec{r}_2'] = 0, \qquad (2.35)$$

and Eq. (2.34) becomes

$$M_1\vec{r}_1 + M_2\vec{r}_2 = (M_1 + M_2)\vec{r}_C. \qquad (2.36)$$

In a short time, Δt, the position vectors will, in general, change by a small amount. These changes are shown in Fig. 2.21. Both $\vec{r}_1$ and $\vec{r}_1'$ change during the interval Δt, but the change in $\vec{r}_1$ does not equal the change in $\vec{r}_1'$ since these vectors are in two different reference frames in relative motion.

During the time interval Δt,

$\vec{r}_1$ becomes $(\vec{r}_1 + \Delta\vec{r}_1)$,
$\vec{r}_2$ becomes $(\vec{r}_2 + \Delta\vec{r}_2)$,
$\vec{r}_C$ becomes $(\vec{r}_C + \Delta\vec{r}_C)$.

Equation (2.36) is rewritten for this later time as

$$M_1(\vec{r}_1 + \Delta\vec{r}_1) + M_2(\vec{r}_2 + \Delta\vec{r}_2) = (M_1 + M_2)(\vec{r}_C + \Delta\vec{r}_C). \qquad (2.37)$$

By subtracting Eq. (2.36) from Eq. (2.37), we find

$$M_1\Delta\vec{r}_1 + M_2\Delta\vec{r}_2 = (M_1 + M_2)\Delta\vec{r}_C. \qquad (2.38)$$

Divide Eq. (2.38) by Δt to obtain

$$M_1\frac{\Delta\vec{r}_1}{\Delta t} + M_2\frac{\Delta\vec{r}_2}{\Delta t} = (M_1 + M_2)\frac{\Delta\vec{r}_C}{\Delta t},$$

or, in the limit as $\Delta t \to 0$,

$$\Delta\vec{r}_1/\Delta t = \vec{v}_1,\ \Delta\vec{r}_2/\Delta t = \vec{v}_2, \text{ and}$$
$$\Delta\vec{r}_C/\Delta t = \vec{v}_C.$$
$$M_1\vec{v}_1 + M_2\vec{v}_2 = (M_1 + M_2)\vec{v}_C.$$

Therefore,

$$\vec{P}_{tot} = (M_1 + M_2)\vec{v}_C, \qquad (2.39)$$

which is our expression for the total momentum of the two-particle (mass) system shown in Fig. (2.21).

Equation (2.39) follows from Eq. (2.36), and Eq. (2.39) is valid for any reference frame used to describe the position and motion of the center-of-mass point.

In the (0') reference frame, the velocity of the center of mass is zero,

$$\vec{v}_c' = 0,$$

so that in this frame

$$\vec{P}'_{TOT} = M\vec{v}'_c = 0.$$

The total momentum in the center-of-mass frame is zero. During the time interval Δt considered above the displacements of M_1 and M_2 are $\Delta\vec{r}_1'$ and $\Delta\vec{r}_2'$ in the center-of-mass frame. These displacements are shown in Fig. 2.21c. The velocities of M_1 and M_2 in the (0') frame are, in the limit of $\Delta t \to 0$,

$$\vec{v}_1' = \Delta\vec{r}_1'/\Delta t$$
$$\vec{v}_2' = \Delta\vec{r}_2'/\Delta t.$$

The displacements $\Delta\vec{r}_1'$ and $\Delta\vec{r}_2'$ are oppositely directed and they obey the relation

$$M_1\Delta\vec{r}_1' = -M_2\Delta\vec{r}_2',$$

because these displacements do not displace the center of mass in the (0') frame. Therefore,

$$M_1\vec{v}_1' = -M_2\vec{v}_2'.$$

The momenta of the two objects exactly cancel one another in the (0') frame and $\vec{P}'_{TOT} = 0$.

In the above discussion the (0') frame was defined with its origin located at the center-of-mass point. However, all of the above statements concerning displacements and velocities in the (0') frame are also valid in other reference frames in which the center of mass is at rest, $\vec{v}_c' = 0$, even though the origins of such frames may not be located on the center-of-mass point. You recall from previous study of motion that the displacement of a point, or the velocity of a point, is the same in all reference frames which differ from one another by only a fixed constant displacement between their origins. The set of reference frames in which the center of mass is at rest is often referred to as the set of "zero momentum" reference frames because the total momentum of the system is zero in each of these frames.

Several interesting things are expressed by Eq. (2.39).

(1) The total momentum of the system is independent of the internal motions of the system.

By the phrase "internal motion" we mean motion relative to the frame of reference in which the center of mass is at rest. This frame is the (0') frame in our discussion. Internal motion involves time rate of change of $\vec{r}_1'$ and $\vec{r}_2'$ and these do not appear in Eq. (2.39), because of the special property of the center of mass, Eq. (2.35).

There is internal motion in this system because the individual $\Delta\vec{r}_1'$ and $\Delta\vec{r}_2'$ displacement vectors are not zero in the (0') reference frame. The internal motions are always of such a combination that the internal momenta $M_1\vec{v}_1'$ and $M_2\vec{v}_2'$ exactly cancel and make no contribution to the total momentum of the system.

When we say "total momentum" we mean the momentum of the system as a whole, and it may then not be surprising that the internal-motion velocities do not appear in Eq. (2.39). That they do not appear is the case only when the internal motion is defined relative to the center-of-mass (or, zero-momentum) frame of reference.

(2) The total momentum of the system equals the product of the total mass and the velocity of the center of mass.

While the internal motions of the system may be involved and complicated, the motion of the center-of-mass point is representative of the motion of the system as a whole. The observations (1) and (2) above are based upon mathematics and they express, with better precision, our earlier division or separation of motion into motion of the system as a whole and motion within the system in the discussion of Fig. 2.13 and Fig. 2.14.

(3) If the system is isolated from the world outside itself, the center-of-mass point obeys the law of inertia.

If the system is isolated, then the total momentum of the system is conserved. Since the total mass does not change, the velocity of the center of mass must be constant according to Eq. (2.39). Whatever complicated internal motions the system has, its center of mass will continue to move with constant velocity. This last statement is the law of inertia, and it is thus contained within the law of conservation of momentum.

Conclusions (1), (2), and (3) above are general and can be derived formally and quickly from the general definition of the center-of-mass vector of Eq. (2.23),

$$M\vec{r}_c = \sum_{i=1}^{N} m_i\vec{r}_i . \qquad (2.23)$$

During a time Δt, each value of $\vec{r}_i$ changes so that

$$\vec{r}_i \text{ becomes } (\vec{r}_i + \Delta\vec{r}_i)$$

and

$$\vec{r}_c \text{ becomes } (\vec{r}_c + \Delta\vec{r}_c).$$

Then,

$$M(\vec{r}_c + \Delta\vec{r}_c) = \sum_{i=1}^{N} m_i(\vec{r}_i + \Delta\vec{r}_i). \qquad (2.40)$$

Subtract Eq. (2.23) from Eq. (2.40) and obtain

$$M\Delta\vec{r}_c = \sum_{i=1}^{N} m_i(\Delta\vec{r}_i). \qquad (2.41)$$

Divide Eq. (2.41) by Δt.

$$M(\Delta\vec{r}_c/\Delta t) = \sum_{i=1}^{N} m_i(\Delta\vec{r}_i/\Delta t).$$

In the limit as $\Delta t \to 0$,

$$(\Delta\vec{r}_c/\Delta t) \to \vec{v}_c$$

$$(\Delta\vec{r}_i/\Delta t) \to \vec{v}_i .$$

Then

$$M\vec{v}_c = \sum_{i=1}^{N} m_i\vec{v}_i$$

$$M\vec{v}_c = \sum_{i=1}^{N} \vec{p}_i ,$$

where $\vec{p}_i = m\vec{v}_i$ is the momentum of the i'th mass element.

The total linear momentum is

$$\vec{p}_{TOT} = \sum_{i=1}^{N} \vec{p}_i .$$

Therefore,

$$\vec{p}_{TOT} = M\vec{v}_c . \qquad (2.42)$$

Conclusions (1), (2), and (3) hold for Eq. (2.42).

2.8 COLLISIONS

Momentum is a vector quantity. The vector character of momentum may not have been fully apparent in the one-dimensional explosions and collisions discussed earlier. In systems free to move in two or three dimensions, the vector character of the conservation of momentum law is clear.

In this section we extend this discussion to more than one dimension, and we will also consider collisions in which the colliding objects rebound and do not stick together after they collide. The completely inelastic collision discussed earlier is not an uncommon occurrence in nature, but it is a special case. Collisions in which objects move apart after the collision are abundant in nature. In the study of physical or biological processes, we often study the interaction of one particle with another or the interaction of one object, or system, with another. Often these interactions occur as collisions. We say that a collision occurs when two objects, which were initially separated, come together to interact with one another and then, in most cases, separate again.

An understanding of collisions is helpful, and there are some things that the laws of physics can say in general about them. One of these is that if the collision occurs in such a way that the participants of the collision are isolated (or temporarily isolated) from the "outside world," then the total momentum of the system of participants must be conserved.

We will assume that there is a period of time both before and after the collision during which each of the colliding particles moves with a constant definable momentum. The total momentum either before or after the collision will equal the vector sum of the momenta of the colliding particles.

We will also assume that the time duration of the actual collision or interaction between the colliding particles is small and negligible. The image of the collision is that of a brief impact during which each particle involved more or less suddenly changes its motion. This condition upon the time duration of the collision is not necessary if the system involved is well isolated. If the system is not isolated, we can assume that there is a "temporary isolation" of the system from the outside world during the collision providing that the collision time duration is short.

The analysis of such a collision will be only approximately correct; often an approximation is good enough. To say that the collision time is short is to say that there is not enough time during the collision for the "outside world" to change the motion of the system by very much. Look at one particle involved in the collision and examine its change in motion during the collision impact. If there is a coupling with the outside world, the momentum of this particle will be changed partly by the interactions with the outside world as well as by the interactions with the other particle(s) in the system during the collision. The sum of these two changes in momentum equals the resultant change of momentum of that particle, $\Delta\vec{p}$. Then

$$\Delta\vec{p} = \Delta\vec{p}_{coll} + \Delta\vec{p}_{ow}$$

where $\Delta\vec{p}_{coll}$ equals the change in momentum due to intrasystem interactions with the other particle(s) of the system and $\Delta\vec{p}_{ow}$ equals the change in momentum due to the intersystem interactions (with the "outside world"). The assumption of "temporary isolation" used in studying much of collision physics can be stated in terms of these momentum changes as,

$$\Delta\vec{p}_{coll} \gg \Delta\vec{p}_{ow}$$

so that $\Delta\vec{p}_{ow}$ can be neglected.

2.8.1 A Sample Collision.

Consider the collision shown in Figs. 2.22a and 2.22b. Figure 2.22a is a stroboscopic photograph of two hard steel spheres of unequal mass which move toward one another, collide, and separate again.

Figure 2.22b is a scale drawing made from the photograph of Fig. 2.22a. The time intervals between successive positions of each sphere (1 and 2), are equal. The displacement between successive positions of each sphere is then proportional to the velocity of the sphere. Velocity measurements can be taken directly from the draw-

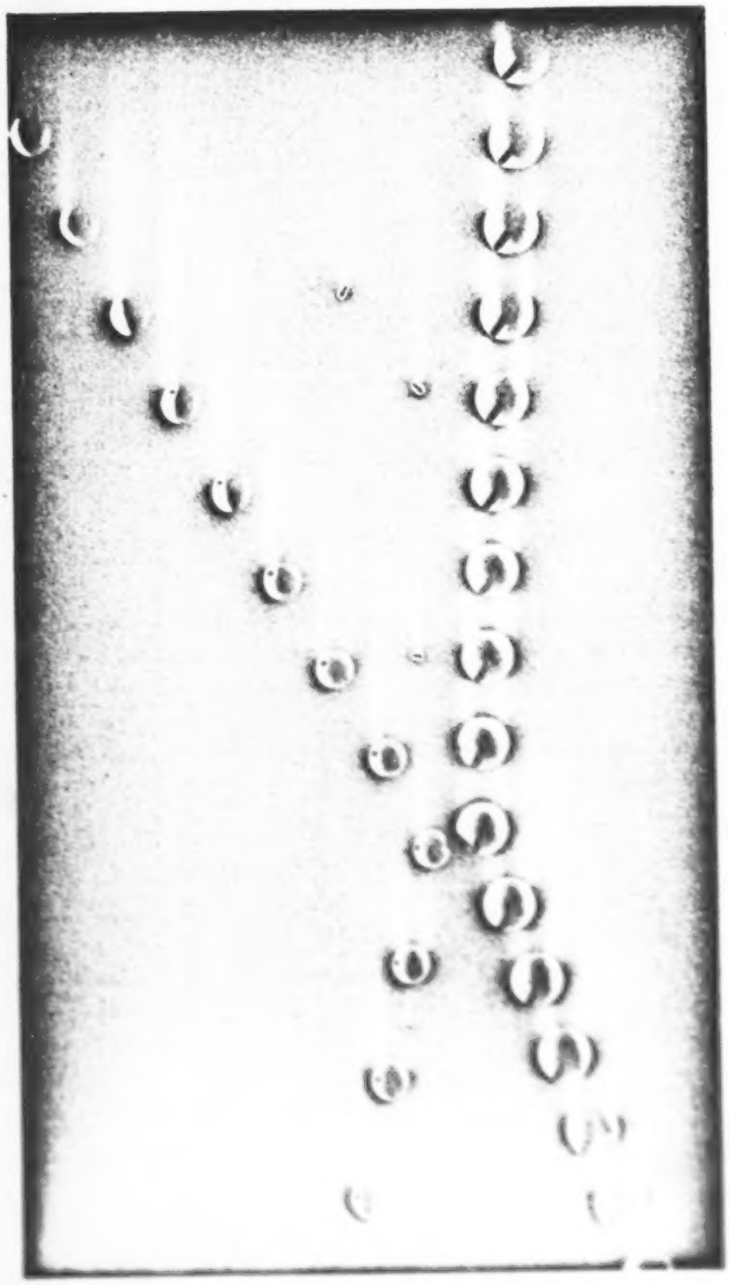


Fig. 2.22a A multiple-flash photograph which shows equal-time intervals in the collision between two spheres of unequal masses. The mass of the large sphere is 201 grams (0.201 kg) and the mass of the small sphere is 85 grams (0.085 kg). Both spheres enter from the top of the photograph. (From PSSC Physics [D. C. Heath and Company, 1960].)

ing proportional to the displacement between successive positions of each sphere.

The spheres shown in Fig. 2.22a are made of hard steel and are suspended from long strings, like pendulum bobs. The two bobs are pulled up and away from their equilibrium positions and released. They are pulled back so far that the strings are almost horizontal when the spheres are released. The speed of the spheres increases as they swing down, but they move at almost constant velocity in the region of the "bottom" of the swing where the collision occurs. We assume that the system consisting of these two spheres is isolated from the outside world insofar as horizontal motion is concerned, even though each sphere is connected to the outside world by its string. In the region of the collision the string is almost vertical and has only a small effect on the horizontal motion. The total momentum of the system is the vector sum of $\vec{p}_1$ and $\vec{p}_2$. In this discussion, the subscript (i) will be used to designate "initial" values of momentum or velocity before the collision. Similarly, the subscript (f)

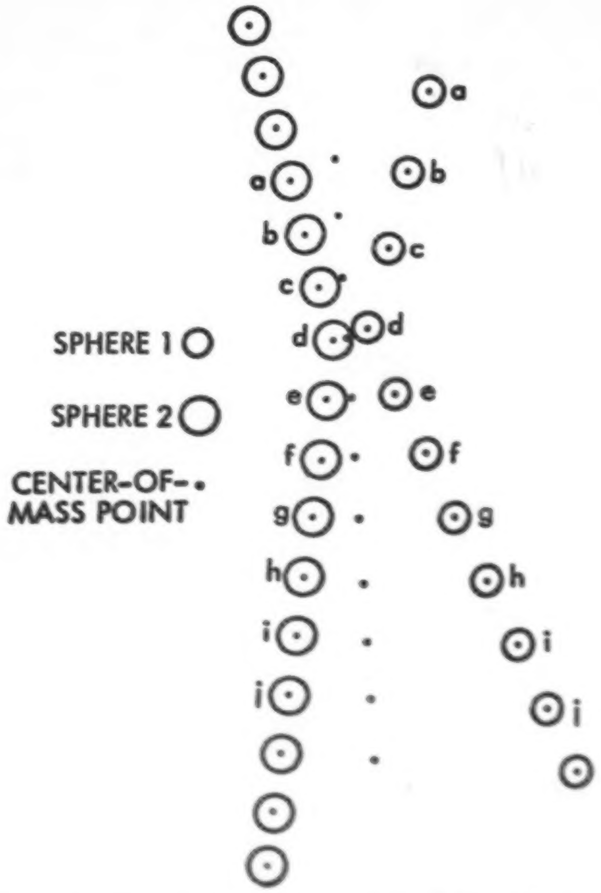


Fig. 2.22b A scale drawing from the photograph of Fig. 2.22a. The center-of-mass point of the two spheres has been determined for each instantaneous position shown in Fig. 2.22a. The constancy of the velocity of the center-of-mass point is a measure of the constancy of the total momentum.

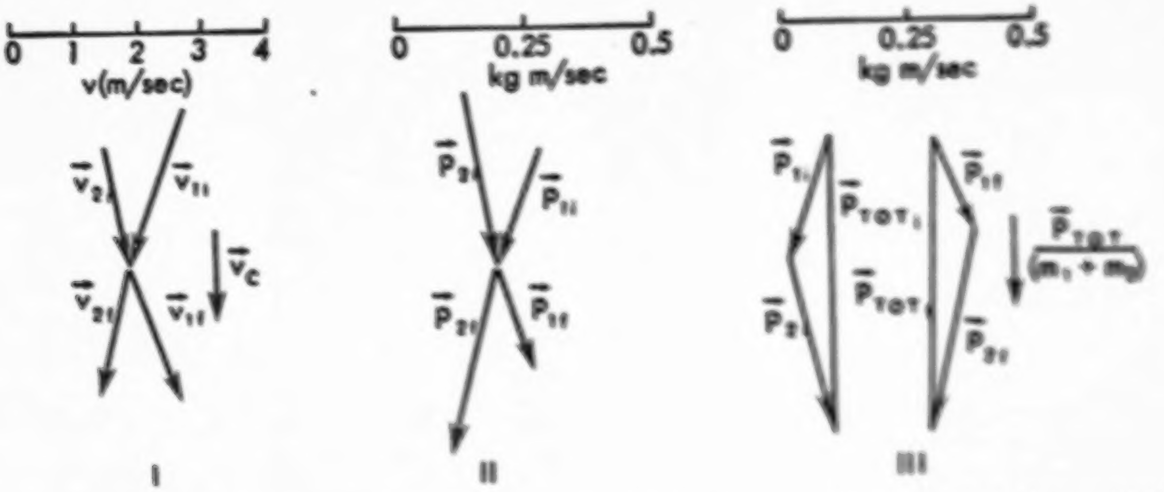

0
1
2
3
4
v(m/sec)
$\overline{v}_{2i}$
$\overline{v}_{1i}$
$\overline{v}_c$
$\overline{v}_{2f}$
$\overline{v}_{1f}$
I
0
0.25
0.5
kg m/sec
$\overline{p}_{2i}$
$\overline{p}_{1i}$
$\overline{p}_{2f}$
$\overline{p}_{1f}$
II
0
0.25
0.5
kg m/sec
$\overline{p}_{1i}$
$\overline{p}_{TOT_i}$
$\overline{p}_{2i}$
$\overline{p}_{TOT_f}$
$\overline{p}_{1f}$
$\overline{p}_{2f}$
$\frac{\overline{p}_{TOT}}{(m_1 + m_2)}$
III

Fig. 2.22c The velocity and momentum vectors found from Fig. 2.22b. (I) the initial velocity vectors of m_1 and m_2, $\overline{v}_{1i}$ and $\overline{v}_{2i}$, and the final velocity vectors $\overline{v}_{1f}$ and $\overline{v}_{2f}$, are shown. The vector $\overline{v}_c$ is the velocity of the center-of-mass point. (II) The initial and final momenta of m_1 and m_2. (III) The (vector) sum of the initial momenta and the sum of the final momenta have been found separately. The value of $\overline{v}_c$ obtained from $\overline{p}_{TOT}$ divided by the total mass is also shown.

will be used for "final" values after the collision.

The initial velocities are seen from the Fig. 2.22 to be almost equal. The exact magnitude of one of the initial velocities is taken as the spacing of subsequent positions of the spheres before the collision and the velocity direction of each sphere is along the line of images. The product of mass and velocity of each disk equals the momentum of each; the initial momenta are shown drawn to scale in Fig. 2.22c, for masses $m_1 = 85$ g, $m_2 = 201$ g.

Since the two objects, 1 and 2, form an isolated system, the final total momentum (after the collision) equals the value of the initial total momentum. Both objects experience momentum changes due to the collision, but the sum of their momenta remains unchanged. Because momentum is a vector, both magnitude and direction of the system's total momentum must remain constant. Let $\overline{p}_{TOT}$ equal the total momentum of this system. Then

$$\overline{p}_{TOT(i)} = \overline{p}_{TOT(f)} . \qquad (2.43)$$

The vector sum of $\overline{p}_{1f}$ and $\overline{p}_{2f}$ is shown in Fig. 2.22b. By comparing their sum, $\overline{p}_{TOT(f)}$, in Fig. 2.22c with $\overline{p}_{TOT(i)}$ in the same figure, the equality expressed in Eq. (2.43) can be checked.

There is another way of expressing the momentum conservation in this collision that will be useful in understanding Newton's third law of mechanics when we encounter it. This way of expressing momentum conservation is

$$\Delta\overline{p}_1 + \Delta\overline{p}_2 = 0, \qquad (2.44)$$

where $\Delta\overline{p}_1$ and $\Delta\overline{p}_2$ are the momentum changes of particles 1 and 2.

The Eq. (2.44) is obtained directly from Eq. (2.43),

$$\overline{p}_{TOT(i)} = \overline{p}_{TOT(f)} , \qquad (2.43)$$

which is, in the form of the particle momenta,

$$\overline{p}_{1i} + \overline{p}_{2i} = \overline{p}_{1f} + \overline{p}_{2f} .$$

Then

$$(\overline{p}_{1f} - \overline{p}_{1i}) + (\overline{p}_{2f} - \overline{p}_{2i}) = 0. \qquad (2.45)$$

Since $\qquad \Delta\overline{p}_1 = (\overline{p}_{1f} - \overline{p}_{1i}), \qquad (2.46)$

and $\qquad \Delta\overline{p}_2 = (\overline{p}_{2f} - \overline{p}_{2i}), \qquad (2.47)$

then, from Eqs. (2.45), (2.46), and (2.47) we obtain Eq. (2.44).

$$\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$$

or,

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 . \qquad (2.48)$$

We have written Eq. (2.44) in the form shown in Eq. (2.48), to emphasize the fact that the changes of momentum of particles 1 and 2 are equal and opposite. During the collision, particle 2 changes the momentum of particle 1, and particle 2 reacts with a change in momentum equal in magnitude to the change of particle 1 but in the opposite direction. An important feature of Eq. (2.48) is that it is true for any inertial frame in which the momentum of the system is expressed. The value of the total momentum depends upon the reference frame in which it is expressed, but the change in momentum of each particle has the same value in all inertial frames which differ only by a constant velocity.

Figure 2.23 shows graphically Eqs. (2.46) and (2.47) for the laboratory frame which is the reference frame of Fig. 2.22. The relationship, $\Delta\vec{p}_1 = -\Delta\vec{p}_2$, is clear in Fig. 2.23.

In any inertial frame (0') moving at constant velocity $\vec{u}$ relative to the laboratory reference frame (0), the momentum of a particle can be found by using the velocity transformation rule, Eq. (1.11) of Chapter 1. The velocity of particle 1 relative to the (0') frame is given by

$$\vec{v}_1{}' = \vec{v}_1 - \vec{u}.$$

Multiply by the mass of particle 1, m_1, and find

$$m_1\vec{v}_1{}' = m_1\vec{v}_1 - m_1\vec{u}$$

or

$$\vec{p}_1{}' = \vec{p}_1 - m_1\vec{u}. \qquad (2.49)$$

The change in momentum of particle 1 in the (0') frame is the vector difference between final and initial momenta in that frame.

$$\Delta\vec{p}_1{}' = \vec{p}_{1f}{}' - \vec{p}_{1i}{}'. \qquad (2.50)$$

By using Eq. (2.49), Eq. (2.50) is

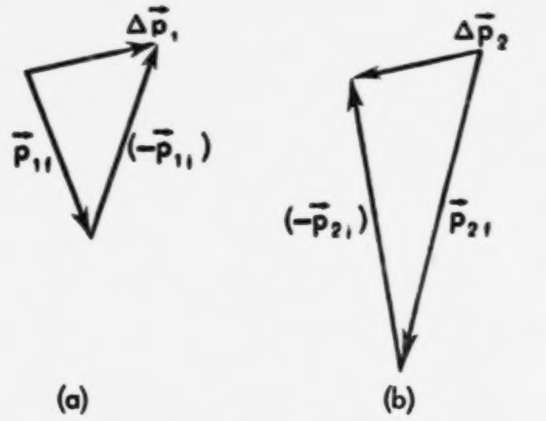


Fig. 2.23 (a) The change in momentum of particle 1 obtained from values of $\vec{p}_{1f}$ and $\vec{p}_{1i}$ in the laboratory frame shown in Fig. 2.22c. (b) The change in momentum of particle 2 for the same reference frame. The momentum changes $\Delta\vec{p}_1$ and $\Delta\vec{p}_2$ are equal in magnitude and opposite in direction.

$$\Delta\vec{p}_1{}' = (\vec{p}_{1f} - m_1\vec{u}) - (\vec{p}_{1i} - m_1\vec{u})$$

$$\Delta\vec{p}_1{}' = (\vec{p}_{1f} - \vec{p}_{1i})$$

$$\Delta\vec{p}_1{}' = \Delta\vec{p}_1 .$$

We have shown that the change of momentum of particle 1 in the (0') reference frame equals the change of particle 1 in the (0) frame. This equality would hold for particle 2 as well,

$$\Delta\vec{p}_2{}' = \Delta\vec{p}_2 .$$

Therefore, since

$$\Delta\vec{p}_1 = -\Delta\vec{p}_2 ,$$

then in the (0') frame,

$$\Delta\vec{p}_1{}' = -\Delta\vec{p}_2{}'.$$

To summarize, the changes in momentum of the two particles are equal and opposite in all inertial frames. The amount by which the momentum of any one particle changes has the same value (magnitude and direction), in all inertial frames.

In section 2.8.2 we will find the complete description of this sample collision in the center-of-mass reference frame. The values of $\Delta\vec{p}_1{}'$ and $\Delta\vec{p}_2{}'$ in this reference frame can then be compared with their values

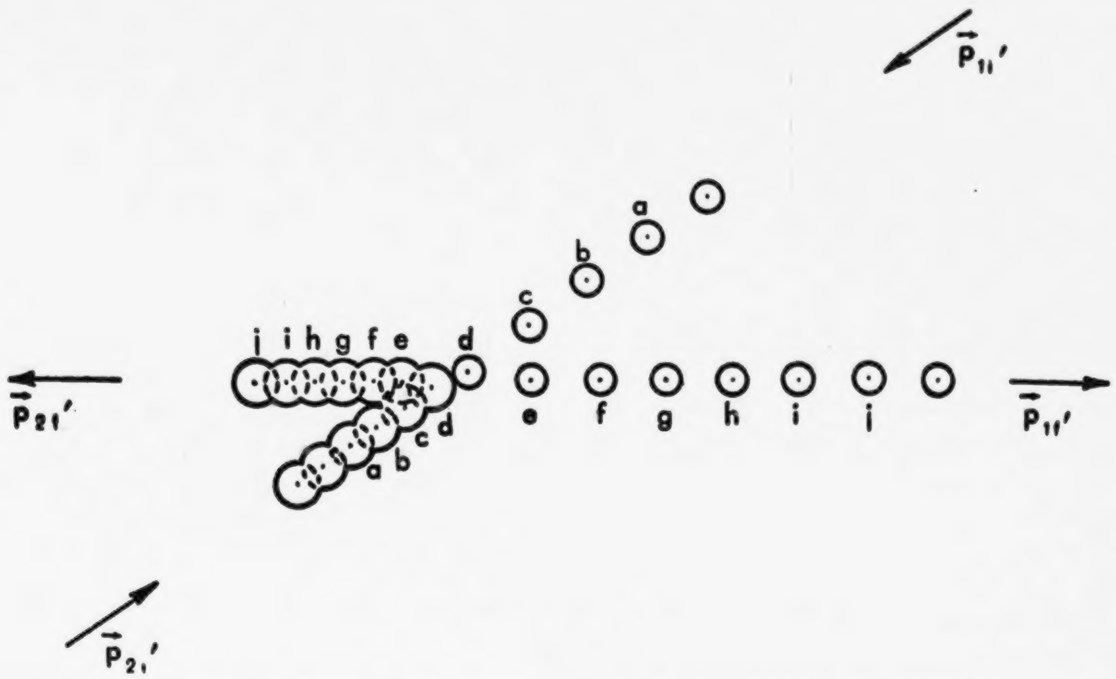


Fig. 2.24 The collision as it appears in the center-of-mass (O') frame of reference. The pairs of letters a-a, b-b, etc., correspond to these pairs of letters exactly as they appear in Fig. 2.22b. This figure is drawn to the same scale as Fig. 2.22b. The vectors representing the momenta have been drawn at twice the length they would have if the scale of Fig. 2.22c were used. The magnitude of each final momentum is about 12% less than the magnitude of each initial momentum in this collision. In comparing the changes of momentum of each sphere in this reference frame, found from $\vec{p}_f'$ and $\vec{p}_i'$, with the changes of momentum shown in Fig. 2.23, keep in mind that there is a factor of two difference in scale of these two figures.

in the laboratory frame as shown in Fig. 2.23.

2.8.2 The Sample Collision in the Center-of-Mass Reference Frame.

The description of collisions is often simpler in the center-of-mass reference frame. Choose the moving frame (O') so that it moves with the center of mass. We can place the origin of the (O') frame at the center of mass of the two particle system. Therefore choose

$$\vec{u} = \vec{v}_c,$$

where $\vec{v}_c$ is the velocity of the center of mass point of our two particles in Fig. 2.22.

Figure 2.24 shows how this collision looks in the center-of-mass frame. The system has a total momentum equal to zero in this reference frame. Notice that the momentum of particle 1 is antiparallel to the momentum of particle 2 in both the initial and final states of this system. (Since the collision is not "head on," the initial trajectories do not necessarily lie along one straight line even though they are parallel.) The sum $(p_1' + p_2')$ is zero both before and after the collision.

The diagrams for vector addition in this reference frame are certainly simpler than in the frame of Fig. 2.22b. The momentum of particle 1 is related to the momentum of particle 2 by a simple change in sign. There are other features of this representation that make it attractive in understanding and analysis. Before getting to them, let us see how to transform the description of this collision from the laboratory frame to the center-of-mass frame.

How is the velocity of this frame, $\vec{v}_c$, determined from a knowledge of $\vec{p}_{1i}$ and $\vec{p}_{2i}$? The most direct way to obtain $\vec{v}_c$ is to make use of the fact developed in section 2.7 that the total momentum of the system equals the product of the total mass of the system and the velocity of the center-of-mass point. Then,

$$(m_1 + m_2)\vec{v}_c = \vec{p}_{1i} + \vec{p}_{2i}$$

and

$$\vec{v}_c = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{(m_1 + m_2)}, \qquad (2.51)$$

where m_1 and m_2 are the masses of particles 1 and 2, respectively.

Equation (2.51) could also be written in terms of the final particle momenta of the system since $\vec{p}_{TOT}$ is constant.

$$\vec{v}_c = \frac{\vec{p}_{1f} + \vec{p}_{2f}}{(m_1 + m_2)} \qquad (2.52)$$

When $\vec{v}_c$ is obtained from the total momentum of the system as in Eq. (2.51) or Eq. (2.52), it is often referred to as the velocity of the "center-of-momentum" or the "zero-momentum" reference frame as well as "the center-of-mass" reference frame. We will use the phrases "center-of-mass frame," "center-of-momentum frame," and "zero-momentum frame" interchangeably. (The center-of-momentum concept is especially useful for example if one of the particles is a photon which has momentum even though its rest mass is zero.)

We already have the total momentum of particles 1 and 2 defined in Fig. 2.22c. When $\vec{p}_{TOT}$ is divided by the total mass, $(m_1 + m_2)$, the result is the center-of-mass vector, velocity $\vec{v}_c$, shown in Fig. 2.22c(III).

For convenience, the instantaneous position of the center-of-mass point, C, of particles 1 and 2 have been indicated on Fig. 2.22b. The displacement between subsequent positions of C in Fig. 2.22b is a measure of $\vec{v}_c$ which can readily be compared with $\vec{v}_c$ in Fig. 2.22c(I). Note also that the spacings between subsequent positions of C are equal, or, $\vec{v}_c$ is constant.

The momentum of a particle in the "zero-momentum" frame is obtained from Eq. (2.49) with $\vec{u}$ set equal to $\vec{v}_c$:

$$\vec{p}_1{}' = \vec{p}_1 - m_1\vec{v}_c. \qquad (2.53)$$

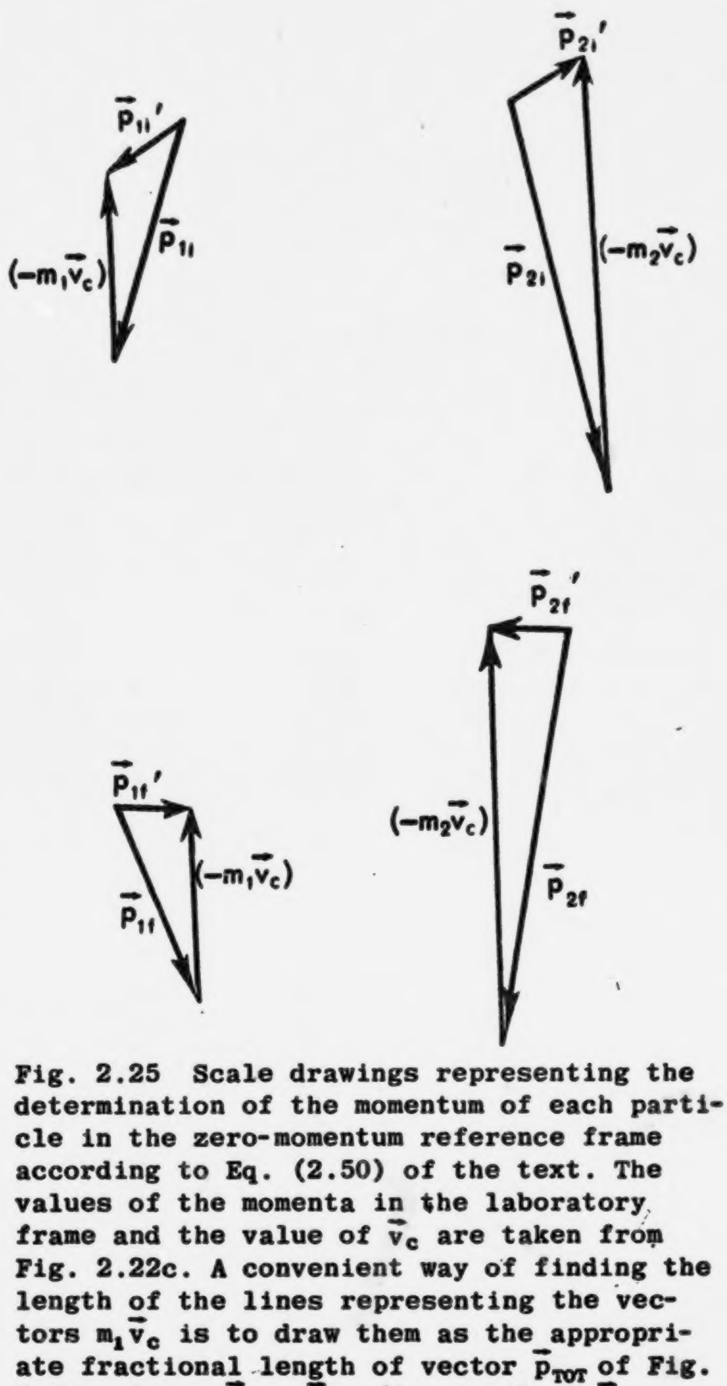


Fig. 2.25 Scale drawings representing the determination of the momentum of each particle in the zero-momentum reference frame according to Eq. (2.50) of the text. The values of the momenta in the laboratory frame and the value of $\vec{v}_c$ are taken from Fig. 2.22c. A convenient way of finding the length of the lines representing the vectors $m_1\vec{v}_c$ is to draw them as the appropriate fractional length of vector $\vec{p}_{TOT}$ of Fig. 2.22c. Since $\vec{v}_c = \vec{p}_{TOT}/(m_1 + m_2)$, $m_1\vec{v}_c$ equals $\vec{p}_{TOT}(m_1/m_1 + m_2)$ and $m_2\vec{v}_c$ equals $\vec{p}_{TOT}(m_2/m_1 + m_2)$. The scale of this drawing is the same as that of Fig. 2.24.

By applying the momentum transformation rule of Eq. (2.53) to each particle, the momentum of each particle in the (0') frame is found from each particle's momentum in the (0) frame. The application of the transformation rule is represented by the vectors, drawn to scale, in Fig. 2.25.

The direction of the vectors drawn in Fig. 2.25 are identical to the directions of the same vectors

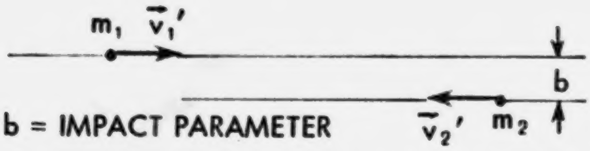


Fig. 2.26 The trajectories of two particles, m_1 and m_2, are shown by dotted lines as they would move if the particles were to continue their initial motions without deflecting one another. The "impact parameter," b, is shown.

as they appeared in Fig. 2.22. The vectors $\vec{p}_{1i}'$, $\vec{p}_{2i}'$, $\vec{p}_{1f}'$ and $\vec{p}_{2f}'$ appear in Fig. 2.24 as they are determined in Fig. 2.25.

The reader can find the vector representing the change in momentum of particle 1 and of particle 2 in the O' frame from the defining relations,

$$\Delta\vec{p}_1' = \vec{p}_{1f}' - \vec{p}_{1i}'$$
$$\Delta\vec{p}_2' = \vec{p}_{2f}' - \vec{p}_{2i}'.$$

Draw or sketch these vector diagrams showing $\Delta\vec{p}_1'$ and $\Delta\vec{p}_2'$. Does $\Delta\vec{p}_1' = -\Delta\vec{p}_2'$, that is, are the momentum changes equal and opposite in this reference frame? Compare $\Delta\vec{p}_1'$ with $\Delta\vec{p}_1$ of Fig. 2.23. Compare $\Delta\vec{p}_2'$ with $\Delta\vec{p}_2$ of Fig. 2.23.

Let us say that the momentum of one particle is known in the zero-momentum frame after the collision. We can quickly find the momentum of the other particle at that same time by simply multiplying the first particle's momentum by minus one.

Notice that the law of conservation of momentum does not provide enough information to allow the unique prediction of both momenta after the collision even though the initial momenta of both particles may be known. There are many (an infinite number of) pairs of vectors representing $\vec{p}'_{1i}$ and $\vec{p}'_{2f}$ that add up to zero total momentum for the system. The question of which of these pairs is the correct one appearing in Fig. 2.24 cannot be answered on the basis of momentum conservation alone.

This limitation on predictability can be demonstrated mathematically. Collisions in a plane are defined by two-dimensional vectors. Two numbers are required to define a two-dimensional vector: its direction (an angle) and its magnitude. If, for example, both final momenta are unknown, there are four unknown numbers, two for each momentum vector. The law of conservation of momentum provides one vector equation that connects these four unknowns with the total momentum which is assumed known. Since this is a vector equation, it must place two conditions on our unknowns. In algebraic form, the vector equation becomes two algebraic equations; for example, as one equation for the x components and one for the y components. These two equations reduce the number of unknowns from four to two. Even a knowledge of what happens to the kinetic energies of the particles (discussed in Chapter 3) would add only one more equation and thus reduce the number of unknowns to one. A unique solution cannot be found from the conservation laws alone.

The missing information has to do with the details of the collision. In principle, the final momenta can be predicted fully if (a) the separation between the two initial trajectories and (b) the nature of the interaction between particles are known.

Sometimes the two objects are spherical; i.e., the interaction between them depends only on the distance separating them and not upon the direction of one from another. Then all that one needs to know about their initial trajectories is the "impact parameter" illustrated in Fig. 2.26. The impact parameter is the distance between the two trajectories that would be obtained if there were no interaction. It would be the distance of closest approach between centers if the two objects continued undoflected. Head-on collisions have zero impact parameter.

We shall not be concerned here with prediction of trajectories of all particles in a collision.

2.8.3 Examples of Momentum Conservation Described in the Laboratory and in the Zero-Momentum Frames of Reference.

Each of the pairs of photographs used in these examples was made by two cameras simultaneously. One camera was fixed in the laboratory and the photographs made by it are labeled (a). The other camera was mounted in a moving frame which traveled at the same velocity as the center of mass of the system being photographed. The photographs made by this camera are labeled (b). A similar camera set-up was used in making Fig. 2.20.

The colliding objects used in making these photographs are dry-ice disks or pucks similar to those described in Section 1 of Chapter 1 with just one exception. Each of the dry-ice pucks used here contains a nine-inch-long bar magnet mounted vertically along its axis. The magnets are mounted so that their "north" ends are down and their "south" ends are up. When the dry-ice pucks are near one another on the horizontal surface on which they glide, the magnets repel one another. Their "interaction" is a magnetic one. These dry-ice pucks collide without actually "touching" each other. The interaction between the magnets causes the momentum of each dry-ice puck to change during a collision or an explosion."

The center of mass of the dry-ice pucks is marked; one is marked by a cross (x) and the other is marked by an open circle (o). The collisions can be considered to be collisions between point particles located at the marked positions. The photographs show instantaneous positions of the dry-ice pucks taken with successive light flashes equally spaced in time.

Look at each of these photographs, with these questions in mind.

(1) In the (a) photographs, can you locate the center of mass for each subsequent position of the dry-ice pucks?

(2) Can you predict what the (b) photograph should look like from the (a) photograph and a knowledge of the mass ratio?

(3) Is the velocity of the center of mass in (a) constant? Should it be?

(4) Is the momentum of this 2-particle system conserved?

(5) Is the change of momentum of one particle equal and opposite to the change of momentum of the other particle in each photograph?

Example 1. Collision Between Two Objects of Equal Mass.

The collision between two magnetic dry-ice pucks of equal mass is shown in Fig. 2.27. In this collision one dry-ice puck (1) is initially in motion and the second (2) is initially at rest in the laboratory frame of reference. The Fig. 2.27a is from the laboratory camera and the Fig. 2.27b shows the collision in the zero-momentum frame.

In Fig. 2.27a, mass (1) initially approaches mass (2) from the left. During the collision (1) forces (2) away and (1) reacts by being deflected from its original trajectory. After the collision they both move to the right while (1) also moves toward the lower edge of the picture and (2) moves toward the upper edge.

In Fig. 2.27b, in the center-of-mass reference frame, the masses move toward one another with (1) coming from the left and (2) coming from the right. After the collision, (1) is moving toward the bottom of the photograph and (2) is moving toward the top.

Figure 2.28a and b show the vectors representing the momenta of (1) and (2) of Fig. 2.27a and b. Since (2) is initially at rest in the laboratory frame its momentum in that frame

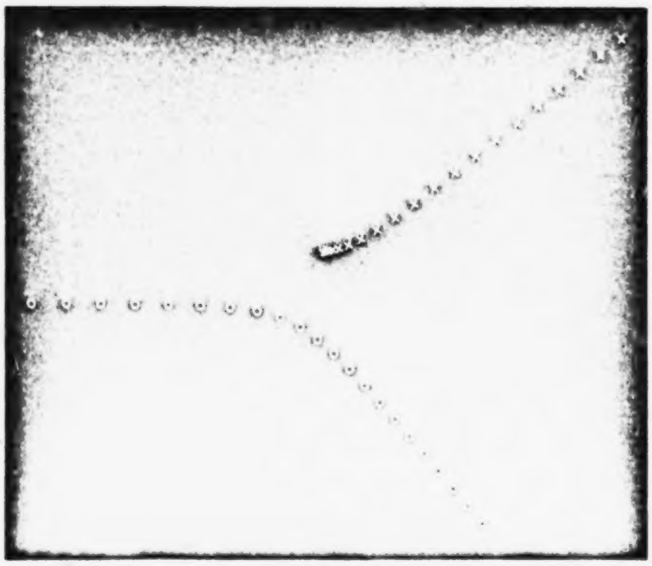


Fig. 2.27a Multiflash photograph of collision between two magnetic dry-ice pucks of equal mass. One mass, marked (x) is initially at rest and the other, (o) is initially in motion from the left. The photograph was taken in the laboratory frame of reference. (Courtesy Film Studio, Educational Services, Incorporated.)

is zero. Let $\vec{p}_{1i} = \vec{p}_0$ in the lab frame. Then the total momentum in this frame is

$$\vec{p}_{TOT} = \vec{p}_{1i} + \vec{p}_{2i}$$
$$\vec{p}_{TOT} = \vec{p}_0 .$$

The vectors $\vec{p}_{1f}$ and $\vec{p}_{2f}$ shown in Fig. 2.28a are drawn from velocity measurements made from Fig. 2.27a. The vector sum of $\vec{p}_{1f}$ and $\vec{p}_{2f}$ equals $\vec{p}_0$ within a two percent accuracy and momentum is conserved.

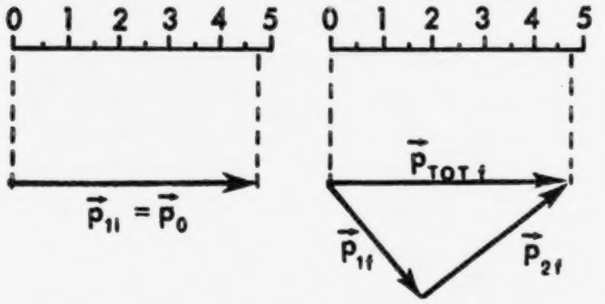


Fig. 2.28a The momentum vectors representing initial and final momenta of pucks (o), $\vec{p}_1$, and (x), $\vec{p}_2$, of the collision of Fig. 2.27a. The momentum of each puck is shown for the laboratory frame of reference. Comparison of the resultant $\vec{p}_{1f} + \vec{p}_{2f}$ and $\vec{p}_{1i}$ is used to confirm the conservation of momentum.

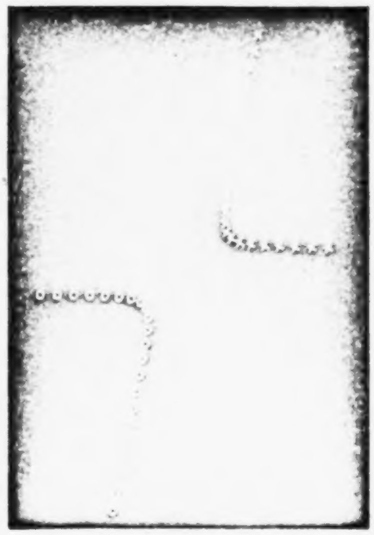


Fig. 2.27b A multiflash photograph of the same collision shown in Fig. 2.27a. This photograph was made simultaneously with Fig. 2.27a by a camera moving in the zero-momentum (center-of-mass) frame of reference. There is one-to-one correspondence between the positions of the pucks in this figure and in Fig. 2.27a. (Courtesy Film Studio, Educational Services, Incorporated.)

The initial and final momenta of particles 1 and 2 are shown in Fig. 2.28b in the zero-momentum frame. The total momentum can be seen to be zero in this frame. Prove to yourself that the value of $\vec{p}\,'_{1i}$ and $\vec{p}\,'_{2i}$ should be $(\vec{p}_0/2)$ and $(-\vec{p}_0/2)$, respectively, in this reference frame.

Example 2. Collision Between Two Objects of Unequal Mass, Mass Ratio Two to One.

The collision between two magnetic dry-ice pucks of unequal mass is shown in Fig. 2.29. In Fig. 2.29a one dry-ice puck (1), mass M_1, initially approaches a second puck (2)

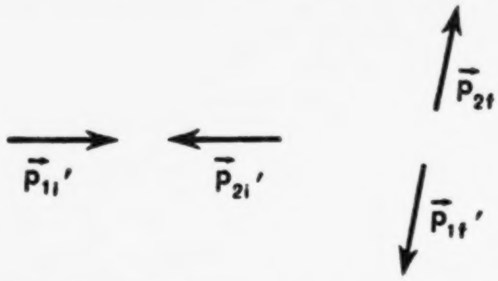


Fig. 2.28b The momentum vectors of the pucks of Fig. 2.27b before and after the collision in the zero-momentum frame. The sum of the momenta is zero both before and after the collision.

Fig. 2.29a Multiflash photograph of a collision of two magnetic dry-ice pucks of unequal mass. One mass, marked (x) is initially at rest. The other dry-ice puck, (o), is initially in motion from the left. The (o) puck has twice the mass of the (x) puck. The photograph was taken in the laboratory frame of reference. (Courtesy Film Studio, Educational Services, Incorporated.)

Fig. 2.29b A multiflash photograph of the same collision shown in Fig. 2.29a. This photograph was made simultaneously with the one used in Fig. 2.29a by a camera moving in the zero-momentum (center-of-mass) frame of reference. Each puck position in this figure corresponds to a puck position shown in Fig. 2.29a. The initial motion is from the left and the right. (Courtesy Film Studio, Educational Services, Incorporated.)

which is at rest. The mass of (1), M_1 equals twice the mass of (2), M_2.

The general directions of motion of the magnetic dry-ice pucks in Fig. 2.29 are similar to the directions of motion in Fig. 2.27. The exact differences in their trajectories are due to the different mass ratio of the pucks in the Fig. 2.29.

Figure 2.30a and b show the vectors representing the momenta of (1) and (2) of Fig. 2.29a and b, respectively. Since (2) is initially at rest in the lab frame (Fig. 2.29a) the total momentum of the system equals the initial momentum of (1), $\vec{p}_{1i}$. Let

$$\vec{p}_{1i} = 2\vec{p}_0$$

so that

$$\vec{p}_{TOT} = 2\vec{p}_0.$$

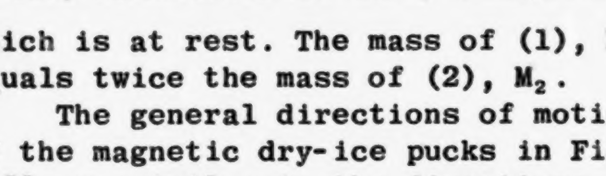


Fig. 2.30a The momentum vectors representing the initial and final momenta of pucks (o), $\vec{p}_1$ and (x), $\vec{p}_2$, of the collision in Fig. 2.29a. The momentum of each puck is shown in the laboratory frame of reference. The resultant of $\vec{p}_{1f} + \vec{p}_{2f}$ can be compared with $\vec{p}_{1i}$ to confirm the conservation of momentum.

The vectors $\vec{p}_{1f}$ and $\vec{p}_{2f}$ shown in Fig. 2.30a are drawn from velocity measurements taken from Fig. 2.29a, and a knowledge of the mass ratio. Compare the vector sum $(\vec{p}_{1f} + \vec{p}_{2f})$ with $2\vec{p}_0$ to see that the momentum is conserved. Note that neither the vector sum of velocities nor an algebraic sum of momenta is conserved. Only the vector sum of momentum is conserved. Momentum is a vector quantity!

In Fig. 2.30b the initial and final momenta of (1) and (2) are shown for the zero-momentum frame. Prove to yourself that the initial values, $\vec{p}_{1i}{}'$ and $\vec{p}_{2i}{}'$, in this reference frame should be $(2\vec{p}_0/3)$ and $(-2\vec{p}_0/3)$, respectively.

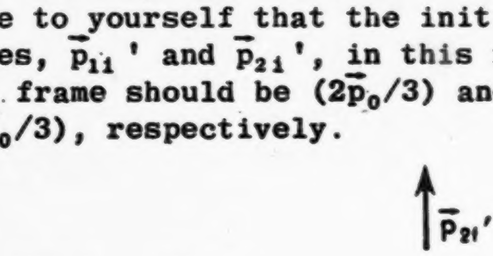


Fig. 2.30b The momentum vectors of the pucks of Fig. 2.29b before and after the collision in the zero-momentum frame. The sum of the momenta is zero both before and after the collision.

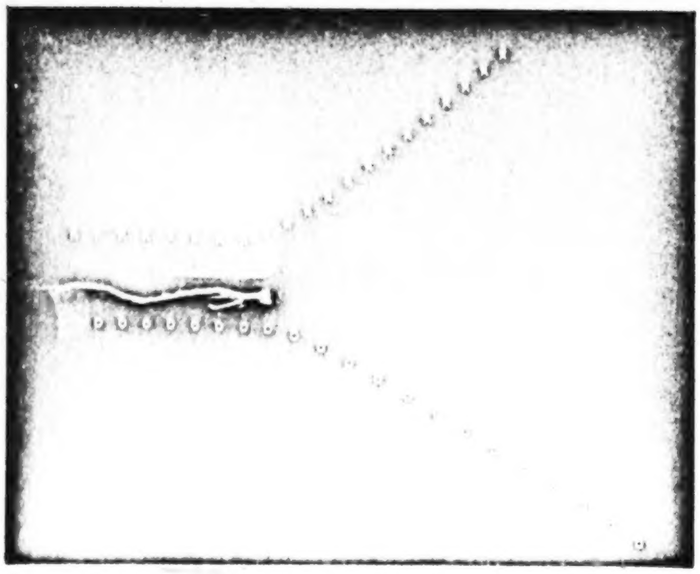


Fig. 2.31a Multiflash photograph of an explosion of two equal mass magnetic dry ice pucks. The "explosion" is triggered by the breaking of a solder wire connection between the two pucks. The solder wire is broken by melting it in a match flame. The wiggly white line is due to the match flame. The coupled pair are initially moving toward the right. The motion is shown in the laboratory frame of reference. (Courtesy Film Studio, Educational Services, Incorporated.)

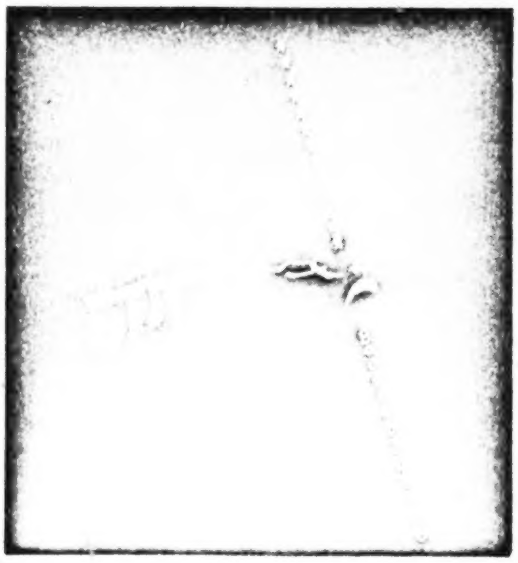


Fig. 2.31b A multiflash photograph of the same explosion shown in Fig. 2.31a. This photograph was taken simultaneously with the one used in Fig. 2.31a by a camera moving in the zero-momentum (center-of-mass) frame of reference. In this frame of reference the pucks are initially at rest and they move away from each other along a straight line after the explosion. (Courtesy Film Studio, Educational Services, Incorporated.)

Example 3. Moving Explosion of Two Objects Having Equal Mass.

The previous moving explosion considered earlier was for one dimensional motion. In this example, the line along which the two-particle explosion occurs is not parallel to the initial motion of the system in the laboratory reference frame.

The explosion is caused by the repulsive force between two magnetic dry-ice pucks. Initially the two objects are held together by a thin solder metal wire. This metal melts at a relatively low temperature and it will melt in the flame from a match. In performing the experiment, the two magnetic pucks are first tied together by a loop of solder wire. The pair is given its initial velocity along the table, and then the experimenter brings a flame from a match to the wire solder. When the solder melts, the "explosion" occurs in that the two magnetic pucks force each other apart. The use of the match flame as the trigger for the explosion is a negligible perturbation on the system and the system can be considered to be isolated.

The explosion of two equal mass pucks is shown in Fig. 2.31a and b. The wiggly white line in the photographs is due to motion of the match flame used. In Fig. 2.31a, the coupled pair of dry-ice pucks initially move toward the right. In the center-of-mass reference frame, shown in Fig. 2.31b, the dry-ice pucks are initially at rest and move away from one another along a straight line after the explosion.

Let the initial momentum of each dry-ice puck in the pair equal $\vec{p}_0$. The total momentum of this system in the laboratory frame is then

$$\vec{p}_{TOT} = \vec{p}_{1i} + \vec{p}_{2i}$$
$$\vec{p}_{TOT} = 2\vec{p}_0 .$$

Fig. 2.32a shows the initial and final momenta of the dry-ice pucks of Fig. 2.31a as drawn from measurements

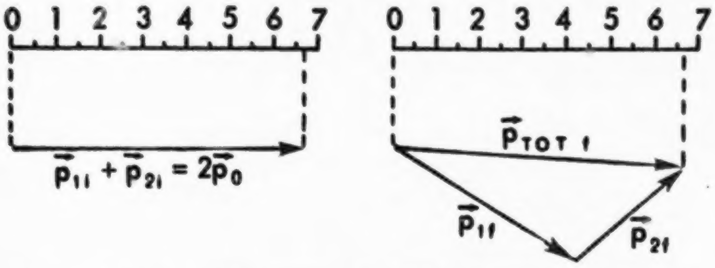


Fig. 2.32a The momentum vectors representing the initial and final momenta of the pucks (o), $\vec{p}_1$, and (x), $\vec{p}_2$, in the moving explosion of Fig. 2.31a. The momentum of each puck is shown for the laboratory reference frame.

of velocity taken from Fig. 2.31. The vector sum of final momenta shows the conservation of momentum at a value of $2\vec{p}_0$, to within experimental accuracy.

Since the initial momentum of each dry-ice puck is zero in the zero momentum frame, only the final momenta appear in Fig. 2.32b. That the vector sum of the final momenta of the two pucks $\vec{p}_{1f}'$ and $\vec{p}_{2f}'$ is zero can be seen in Fig. 2.32b.

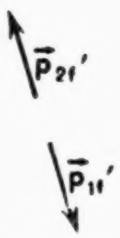


Fig. 2.32b The momentum vectors of the pucks of Fig. 2.31b after the explosion in the zero-momentum frame of reference. The pucks are at rest before the explosion in this reference frame.

Example 4. Moving Explosion of Two Objects Having Unequal Mass, Mass Ratio Two to One.

Figure 2.33a shows the two magnetic dry-ice pucks move in from the left and explode apart near the center of the photograph. Clearly the more massive dry-ice puck acquires a lower velocity perpendicular to the original direction of motion than does the less massive one. We let m_1 equal $2m_2$ and draw the momentum vectors for this explosion from velocity measurements on the photographs. Figure 2.33b shows the explosion in the center-of-mass frame of reference.

In Fig. 2.34a, the initial and final momenta are drawn to scale for the explosion in the laboratory frame.

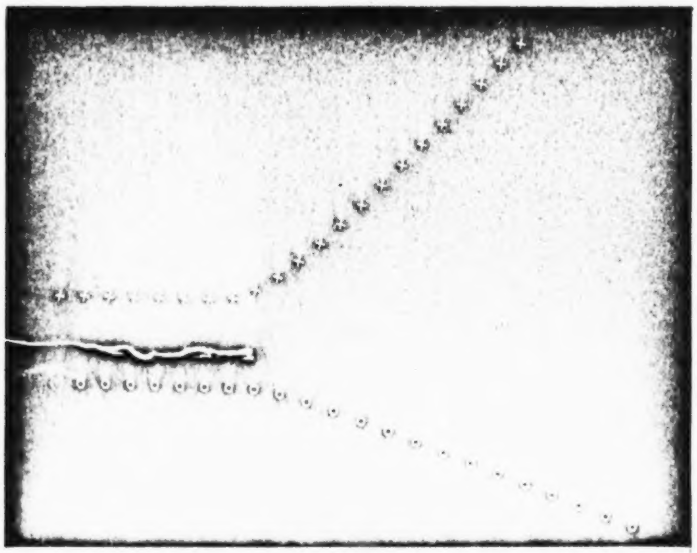


Fig. 2.33a Multiflash photograph of two dry-ice pucks of unequal mass. The mass ratio is two. The coupled pair of pucks is initially moving toward the right in the photograph. The puck marked (o) has twice the mass of the puck marked (x). The motion is shown in the laboratory frame of reference. (Courtesy Film Studio, Educational Services, Incorporated.)

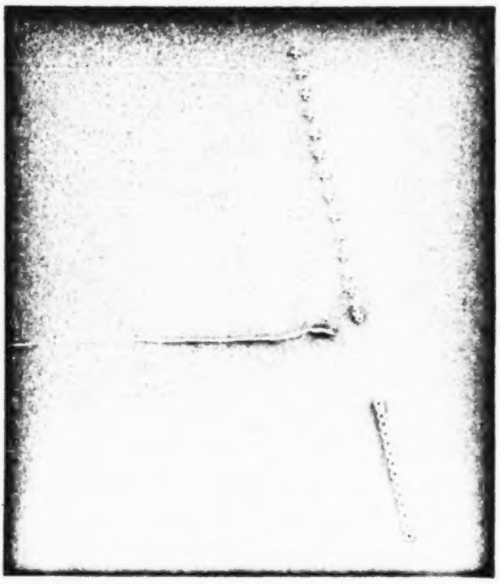


Fig. 2.33b A multiflash photograph of the same explosion shown in Fig. 2.33a. This photograph was taken simultaneously with the one used in Fig. 2.33a by a camera moving in the zero-momentum (center-of-mass) frame of reference. In this frame of reference the pucks are initially at rest and they move away from each other after the explosion in a straight line. (Courtesy Film Studio, Educational Services, Incorporated.)

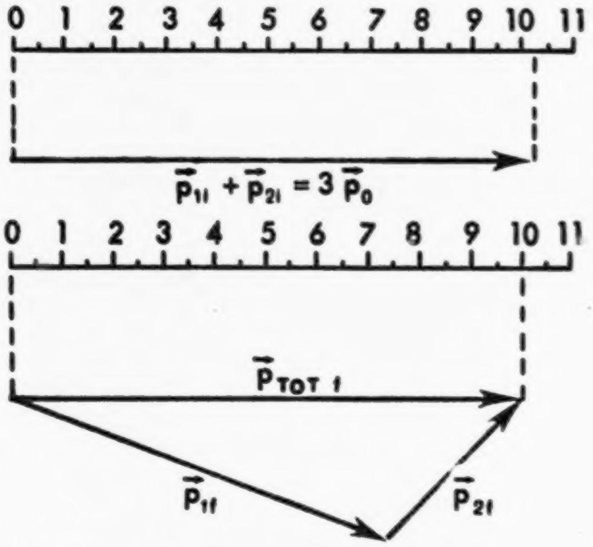


Fig. 2.34a The momentum vectors representing the initial and final momenta of the pucks (o) and (x) in the moving explosion of Fig. 2.33a. The momentum vectors are shown for the laboratory frame of reference.

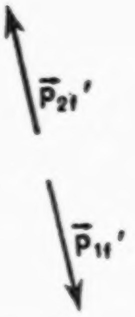


Fig. 2.34b The momentum vectors of the pucks of Fig. 2.33b after the explosion in the zero-momentum frame of reference. The pucks are at rest before the explosion in this reference frame.

The total momentum is

$$\vec{p}_{TOT} = 3m_2\,\vec{v}_0 = 3\vec{p}_0.$$

where $\vec{v}_0$ is the initial velocity obtained from the photograph. The vector sum of the final momenta equals $3\vec{p}_0$.

In the center-of-mass frame, the final velocities are in ratio two to one so that the momenta are equal and opposite in this reference frame. The vectors representing $\vec{p}_{1f}{}'$ and $\vec{p}_{2f}{}'$ in the center-of-mass frame are shown in Fig. 2.34b.

If you were given $\vec{p}_{1f}$ and $\vec{p}_{TOT}$ for this explosion could you find the momentum in the center of mass frame, $\vec{p}_{1f}{}'$?

These examples have demonstrated that the law of conservation of momentum works only if we treat momentum as a vector quantity. The rules we follow in finding sums and differences are the rules of vector algebra. We see that physical systems demonstrate these rules. There is something marvelous about this unity of mathematics and physics. If you don't begin to see this unity and understand some examples of it, then you will not have seen physics. When physics problems get solved they are first expressed in a mathematical form. The transformation of physics problems into mathematical problems is often the toughest part of physics problem solving. Once this is properly done, the solution to the ensuing mathematical problem, found by the logic of the mathematics, will be the answer to the physics problem!

As we study physics further, we will see further physics examples and problems. Without becoming expert in problem solving, it is hoped that there will be sufficient exposure to grant some insight into the unity of physics and mathematics.

3 THE CONSERVATION OF ENERGY

3.1 THE PRINCIPLE OF ENERGY CONSERVATION

Matter in motion has a price! The inertia of mass at rest requires a force to put that mass in motion, and momentum still does not tell the whole story. In the experiments discussed in this book, a variety of agents were used to put matter in motion: a compressed spring, a chemical explosive, another piece of moving matter, and so forth. These agents possess a potentiality for putting matter in motion that differs somehow from the ability of passive agents to change the direction of the vector momentum. Energy, the "price" of matter in motion, differs qualitatively from momentum. An ice skater works very hard to build up speed to perform his tricks, but then he glides around intricate curves and loops with little further effort, while his vector momentum may change rapidly and even reverse. The effort required to make a moving railway train turn a corner and redirect its constant speed is qualitatively different from the ranting and snorting of the engine that accompanies putting it into motion from rest, giving it energy of motion. There is something called energy released in the burning of the fuel that is needed to put the train or the skater in motion, but which is not required to change their directions. *Energy* is a measure of the price of mass in motion.

A moving mass itself can, in a collision, transfer motion to another mass while leaving the rest of the world unaffected, and one form of energy is present in the motion itself. This form of energy is called *kinetic energy*. Because there are many systems which do not involve obvious motion, but which contain the potentiality for producing it, a description of the physical world must include other forms of energy as well. The usefulness of the concept of energy stems from the fact that we can define each of these forms of energy in a measurable way, such that *the total amount of energy in the universe never changes*. Whenever an experiment has been performed with careful bookkeeping of all the different forms of energy, it has been found that the total amount of energy remains constant. Energy that is removed from one form appears in others, and energy that appears in one form is always balanced by energy that disappears in others. It is in this sense that physicists associate their formulas for many different forms of energy. A compressed spring, a nuclear reactor, a gasoline-air mixture waiting to burn do not share a common appearance or feel or sound. They do not appear at all alike to the senses, and the different forms of energy are very elusive, but there is always a strict correspondence in the amount of one kind of energy that is exchanged for a given amount of another.

The great variety of forms in which energy exists is suggested by the number of methods one might use to put a mass in motion. The mass could be struck with another moving mass, or it could be accelerated by a stretched elastic band. It could be put into motion by tying to it one end of a rope that is wound at the other end around the shaft of an electric motor, or accelerated from rest simply by the pull of the earth. A rush of hot gases from the burning of gasoline, the steam from heated water, or a hurricane born in the tropical oceans can start the mass moving. In many cases it is fairly easy to trace the history of the energy given the moving mass even farther back. The electric motor that accelerates the mass may be connected by wires to a generator driven by a nuclear reactor many miles away. The living things from which the gasoline was ultimately derived took energy from the brilliant rays of the sun. Fortunately many kinds of energy in this bewildering array have features in common, and the different forms of energy can be grouped into classes and subclasses that simplify the over-all picture.

The most convenient classification of energies for our purpose is in terms of the kinds of quantities needed to calculate them. In some cases these classifications will be rough and will overlap a little, but they are still useful. The kinetic energy of matter in motion depends only on the mass involved and the speed with which it is moving. If a mass m moves with velocity v, its kinetic energy KE depends on m and v (the size of v) according to the equation,

$$KE = \tfrac{1}{2}mv^2 . \qquad (3.1)$$

In the mks system of units (meter-kilogram-second) this equation gives the energy in units called Joules.

Other kinds of energy, which can be converted to kinetic energy, depend only upon a position. The name potential energy for this form probably arose because it has the potentiality of being converted to the more obvious kinetic energy. The elastic potential energy of a stretched or compressed spring depends upon its length, the position of one end relative to the other. Opposite charges on the top and bottom of a thundercloud have a great attraction for each other. The electrical potential energy which depends upon their separation is converted to a rushing kinetic energy when the charges flow together in a lightning stroke, and then to light energy, heat energy, sound energy, and chemical energy in the subsequent collisions with intervening air molecules. Because the forces between atoms themselves are basically electrical in nature, many forms of potential energy are really kinds of electrical potential energy. When the bonds between atoms are slightly distorted, as in stretching a spring, the change in electrical potential energy is sometimes called elastic potential energy. When new electrical bonds between atoms are created and old ones destroyed, as in the burning of gasoline, the change in electrical potential energy is included as chemical energy.

The gravitational potential energy associated with a rock separated from the surface of the earth can be converted to kinetic energy by letting rock and earth come together. If the rock falls only a few hundred feet, the kinetic energy is most apparent in the motion of the rock; but if it falls for thousands of miles, much of the energy will be used in heating and vaporizing the rock as a white-hot shooting star flashes across the sky. Any two objects have a gravitational potential energy that depends only on their masses and their separation. This statement applies not only to the earth and another object, but to two asteroids, two dust particles, or two atoms isolated in space. At the surface of the earth gravitational interactions are very important. Between two dust particles the gravitational potential energy is very small, but then so are the dust particles themselves. The gravitational potential energy has an important effect on the future of a cloud of countless billions of atoms and dust particles as it collapses into a hot (energetic) ball of fire called a star. However, in assembling a proton and an electron into one hydrogen atom, the change of gravitational potential energy is so small compared with the changes in electrical energy that it can always be neglected. For masses typical of elementary particles like the electron and proton, and for the amount of electric charge on these particles (the smallest nonzero electric charge), the electrical energy changes completely dominate the gravitational energy. Only for electrically neutral matter (with no net electrical charge) is gravitational potential energy significant.

The motions of macroscopic objects can be described in terms of gravitational or electrical forms of potential energy. Even the interactions of electrons and atomic nuclei as they interact to form atoms and molecules involve potential energies that are basically electrical. Of course, describing the wealth of varied phenomena that make up the motions of the stars, the planets, their inhabitants, and the atoms of which they are composed into changes of kinetic and two kinds of potential energy is a terribly incomplete description. But it does reveal a thread that runs through the whole tapestry of natural phenomena.

In laboratory experiments which probe the nucleus of the atom itself, in events which involve distances only one ten-thousandth of an atomic diameter, new forms of potential energy become important. Nuclear potential energies describe the effects of forces which act only over the very short distances found between particles inside the atomic nucleus (less than 10^{-14} meter). They are important to the macroscopic world in that they determine the constituents from which it is made. In this submicroscopic realm the description of nature in terms of forces is very complicated, and the use of the energy concept has been a necessary part of whatever progress has been made in understanding it.

To build a conservation law for energy, the kinetic and potential energies of an object treated as a particle (i.e., specified by the position of a point) are not enough. The motions of the parts must also be taken into account, if energy can be transferred to or from these motions. A walking man has more kinetic energy than just the kinetic energy of his center-of-mass motion. Figure 3.1 does not do justice to the full complexity of a man in motion, but it illustrates the point with a model which is perhaps the next step in complexity from a single mass moving with one velocity.

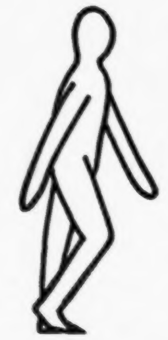


Fig. 3.1a A man walking. This is a very complicated motion which involves more than just one mass moving with a velocity. The next two parts of this figure illustrate just a little of this complexity.

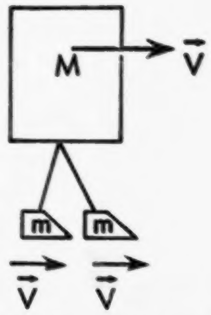


Fig. 3.1b A simple model that can "walk." Here it is sliding. The main mass M and the two feet share the common velocity $\vec{V}$, which is therefore the velocity of the center of mass. The kinetic energy is $\frac{1}{2}(M + 2m)V^2$.

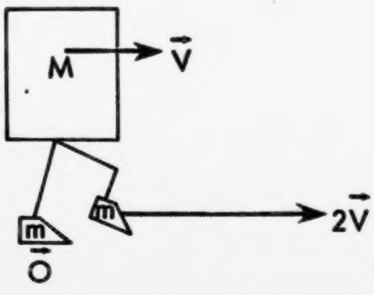
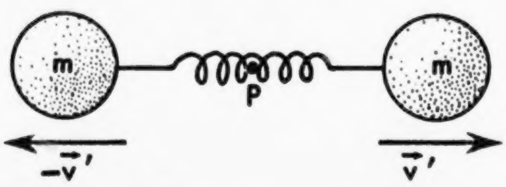


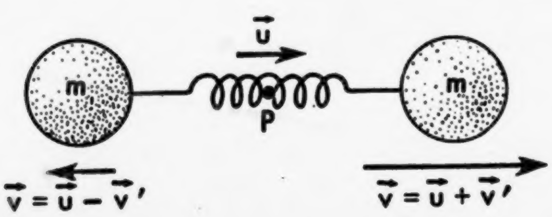
Fig. 3.1c The model "walking." If one foot is at rest on the ground and the other moving forward at velocity $2\vec{V}$, the over-all center of mass still moves with the mean velocity, $\vec{V}$. The total kinetic energy exceeds that in Fig. 3.1b:

$$\begin{aligned} KE &= \tfrac{1}{2}MV^2 + \tfrac{1}{2}m(2V)^2 \\ &= \tfrac{1}{2}(M + 2m)V^2 + mV^2 . \end{aligned}$$

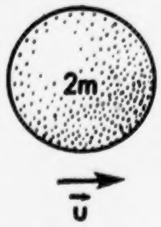
As well as internal kinetic energies associated with their velocities, real objects commonly have internal potential energies associated



(a) Two equal masses connected by a spring of negligible mass. In the inertial frame in which the center of mass P is at rest, the two masses have equal and opposite velocities. A particular situation at one instant of time is illustrated here.


$\vec{u}$
m
P
m
$\vec{v} = \vec{u} - \vec{v}'$
$\vec{v} = \vec{u} + \vec{v}'$

(b) Viewed from another inertial frame, which happens to be our laboratory frame, the center of mass moves with the velocity $\vec{u}$. The masses move in this frame with velocities $(\vec{u} - \vec{v}')$ and $(\vec{u} + \vec{v}')$ at the moment illustrated.



(c) A single mass, 2m, moving with the center-of-mass velocity $\vec{u}$.

Fig. 3.2 The total energy of a vibrating dumbbell from three viewpoints. The energy of the dumbbell seen in the laboratory as in (b) is

$$E = \tfrac{1}{2}m(u - v')^2 + \tfrac{1}{2}m(u + v')^2 + \text{PE of spring}$$
$$= mu^2 + mv'^2 + \text{PE of spring}.$$

This quantity differs from the kinetic energy of a single mass 2m moving with the center-of-mass velocity u, as in (c). The kinetic energy in (c) is

$$KE = \tfrac{1}{2}(2m)u^2 = mu^2.$$

We shall see later that it is more than coincidence that the internal energy (difference between E for (b) and KE of (c) is just the total energy calculated from the viewpoint of (a), in the rest frame of the center of mass:

$$\text{Internal energy} = \tfrac{1}{2}mv'^2 + \tfrac{1}{2}mv'^2 + \text{PE of spring} = mv'^2 + \text{PE of spring}.$$

with the relative positions of their parts. The moving vibrating dumbbell of Fig. 3.2 is made of two equal masses m connected by a spring of negligible mass. The motions of the masses give this object kinetic energy, and the spring stores a potential energy which depends on its length. When the dumbbell participates in processes in which energy is transferred only to or from the motion of the whole (as for example, when it is released to fall under the influence of gravity), then the whole dumbbell can be treated as one particle with a total mass 2m, as in Fig. 3.2c. The energy of the internal motions need not be taken into account when it remains constant. Only when the dumbbell is involved in more complicated situations, where energy can be transferred to the motion of one mass relative to the other, is it necessary to treat the two masses separately. The total energy in the more detailed analysis includes the kinetic energy of a particle of mass 2m moving with the center-of-mass velocity plus the kinetic energies of the two masses m in the center-of-mass reference frame plus the potential energy of the compressed or stretched spring. If the center of mass of the system moves with velocity $\vec{u}$ relative to the labortory, and if the two masses move with velocities $+\vec{v}'$ and $-\vec{v}'$ relative to the center of mass, the total energy E of the moving, vibrating dumbbell in the laboratory reference frame is (see Fig. 3.2),

$$E = \tfrac{1}{2}(2m)u^2 + \tfrac{1}{2}mv'^2 + \tfrac{1}{2}mv'^2 + \text{PE of spring},$$

or

$$E = mu^2 + mv'^2 + \text{PE of spring}.$$

That the separation into center-of-mass energy (Mu^2) plus internal energy (mv'^2 + PE of spring) takes place so naturally will be shown later to be more than a coincidence. When a part of each mass m can change its motion separately from the rest, an even more detailed analysis is required.

Objects of a size we can handle are all made of smaller pieces called atoms, and, in spite of our best efforts, the energy of a large mass in motion has a tendency to become dissipated in the random motions of the atoms that comprise it and its environment. The air pushed aside as an automobile passes, the viscous oil in the bearings, the squashing of the tires, all transfer energy from the macroscopic motion of the car to the random motions of atoms. The conversion of the over-all directed kinetic energy of center-of-mass motion to the energy of random atomic motions we call heat can even be useful. The kinetic energy of an automobile in motion must sometimes be dumped in a hurry. The automobile's brakes use friction to convert its kinetic energy to heat. A typical 1500-kilogram automobile moving at 30 meters per second (about 60 mph), has a kinetic energy of about 7×10^5 Joules according to Eq. (3.1). After its brakes have brought it to a halt, there is no more kinetic energy associated with the car's center-of-mass motion. In searching for a clue as to where the energy went, even an initially uniformed observer would be impressed by the temperature rise of the car's brakes. For the car above, the approximately 30 kilograms of steel in the brake drums and shoes could be warmed by more than 40°C. in one stop, becoming too hot to touch. Further experimentation would reveal that the temperature rise of the brakes increases with the kinetic energy of the car's motion. More quantitative and careful experiments on a variety of moving objects would show that a definite amount of kinetic energy is always required to produce the same thermal effect. (A common unit of thermal energy, the calorie, is the amount of heat required to raise one gram of water one degree centrigrade.) This consistency in the amount of heat obtained from each unit amount of kinetic energy leads to the conclusion that heat is a form of energy. Heat is a variety of internal energy associated with the myriad random motions that go on among the atoms of the hot brake lining or any other material substance.

The heat energy associated with the motions of atoms in a solid brake lining or molecules whizzing about in a gas is too complicated and involves too many particles to keep track of in detail. Fortunately, the total internal energy of such a system is often found to be equally shared, on the average, among all the kinetic and potential energies of its parts. When the many particles in a complicated system interact with each other quickly, so that the particles share their energies in mutual interactions many times before and during the time it takes to make a measurement on the system, the system can be thought of as "homogenized," In such systems each particle has its average share of energy, just as each drop in a quart of homogenized milk has its share of cream. Then the total energy of the system can be simply related to the average energy of each particle. The average particle energy suffices to predict many of the gross properties of the whole system, for example, the pressure of a gas in a container or the length of a solid. A useful measure of the average energy of the particles in these complicated but homogeneous systems is the temperature.

Two identical objects (say, two brake shoes) can be placed in physical contact so that the atoms of one can bump against the atoms of another, exchanging energy. If, on the average, heat energy flows in these collisions from object A to object A', we say that the temperature of A is higher than that of A'. The temperatures of A and A' are equal when, on the average, no energy flow takes place in either direction, as demonstrated by watching some macroscopic property that depends on the average particle energy. When the two objects A and A' are identically constructed, it comes as no great surprise that energy flows from the system with more energy to

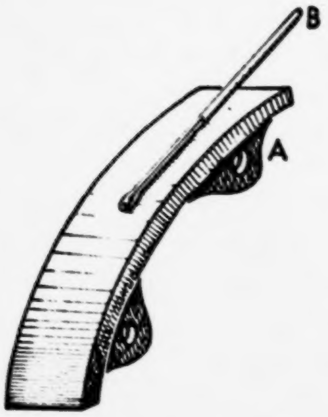


Fig. 3.3 Brake shoe A and thermometer B in physical contact. The brake shoe and the thermometer are at the same temperature when there is no net energy flow into or out of the thermometer, as demonstrated by the fact that the length of the liquid in the capillary tube does not change with time.

that with less; i.e., that a higher temperature for the same system of particles implies a higher total heat energy. The temperature provides a measure of this tendency of heat energy to flow that can also be applied to different objects. Two different objects A and B have the same temperature if, when they are placed in physical contact (for example a brake shoe placed against a thermometer as in Fig. 3.3), no energy flows on the average in either direction.

Energy appears in still another wonderful disguise as mass itself. The kinetic and potential energies of an object, from both its over-all motion and its internal motions, affect its mass. The conversion factor from energy to mass is so small, however, that the mass changes associated with the kinetic energies we find in a macroscopic object are insignificant fractions of the mass of the object at rest. The kinetic energy of a mass moving at typical laboratory speeds is given Eq. (3.1). The mass change Δm associated with any energy change ΔE in a system is given by

$$\Delta m = \frac{\Delta E}{c^2}, \qquad (3.2)$$

where c is the speed of light (3×10^8 meters per second). If a mass m_0 at rest is given a kinetic energy KE, associated with the speed v by Eq. (3.1), the corresponding increase in mass is given by substitution of KE from Eq. (3.1) for ΔE in Eq. (3.2):

$$\Delta m = \frac{\frac{1}{2} m_0 v^2}{c^2}. \qquad (3.3)$$

The fractional mass change ($\Delta m/m_0$) is obtained by dividing both sides of Eq. (3.3) by m:

$$\frac{\Delta m}{m_0} = \frac{1}{2}\frac{v^2}{c} \qquad (3.4)$$

The speed c is so extremely large, compared with any reasonable laboratory speed that the fractional increase in mass is negligible. A speed of 1000 miles per hour is equal to about 500 meters per second in mks units. For this speed the fractional increase in mass is only

$$\frac{\Delta m}{m_0} = \frac{1}{2}\left(\frac{5 \times 10^2}{3 \times 10^8}\right)^2 \cong 1 \times 10^{-12}. \qquad (3.5)$$

It is only with microscopic particles, where speeds can be made comparable with the speed of light, that the fractional mass increase with kinetic energy is significant.

For speeds v comparable to c, Eq. (3.1) is no longer adequate, and must be replaced by an equation for the total energy,

$$E = mc^2 \qquad (3.6)$$

in which the mass m is not the mass of the object at rest, but the mass of the object now, in the situation where we want to know its energy. For an object with mass m_0 at rest, its mass m when moving with speed v is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (3.7)$$

This gives it a total energy when in motion of

$$E = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \ . \qquad (3.8)$$

at rest, its total energy, according to Eq. (3.6) was only

$$E_0 = m_0c^2 . \qquad (3.9)$$

The kinetic energy, which had to be supplied to set it in motion, is

$$KE = E - E_0 = m_0c^2\left(1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\right) . \qquad (3.10)$$

When (v^2/c^2) is very small compared to one, it is a very good approximation to write

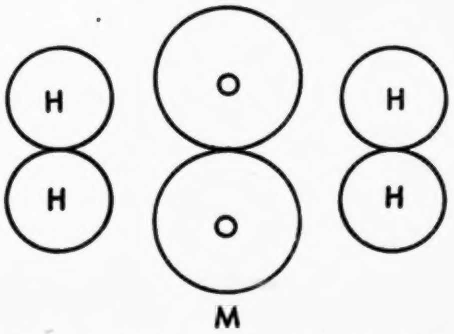
$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \cong 1 + \frac{1}{2}\frac{v^2}{c^2} \ . \qquad (3.11)$$

When this approximation is valid (and v^2/c^2 is certainly small in Eq. (3.5)) the expression (3.10) for the kinetic energy can be written

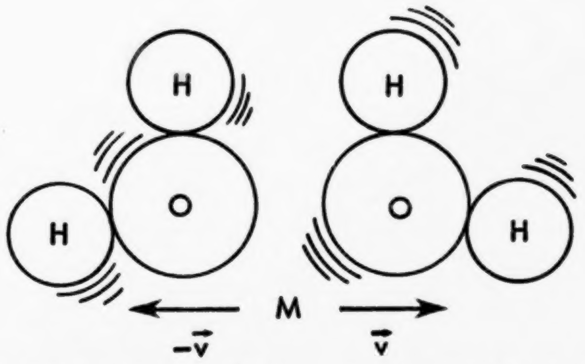
$$KE \cong m_0c^2 \left[1 - \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\right]$$

$$\cong \frac{1}{2}m_0v^2 . \qquad (3.12)$$

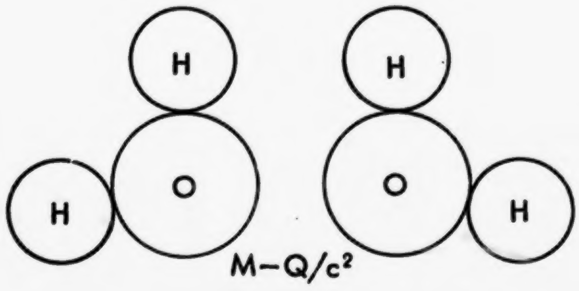
The error in the approximation in Eq. (3.12) is of the size of m_0v^4/c^2, which is smaller by a fraction something like v^2/c^2 than the value $\frac{1}{2}m_0v^2$. Recall Eq. (3.5) for "typical" numbers.



(a) Two H_2 molecules and one O_2 molecule at rest. The total mass is M.



(b) After the reaction the two moving jiggling H_2O molecules still have mass M if they retain the energy Q released by the reaction as energy of internal motion.



(c) If the energy Q released by the reaction is removed, the mass of the products at rest is smaller by the amount Q/c^2 than the mass M of the reactants.

Fig. 3.4 In the reaction $2H_2 + O_2 = 2H_2O$ the mass decrease is about 2×10^{-36} kg, out of a total mass M of 6×10^{-26} kg, a fractional decrease of 3×10^{-11}.

Not only kinetic energy changes but also other kinds of energy changes affect the mass of an object. The kinetic and potential energy changes in the chemical reaction

$$2H_2 + O_2 = 2H_2O$$

are such that the internal energy of the two water molecules is less than that of the oxygen and two hydrogen molecules by about 2×10^{-19} Joules. If this energy remains in the system as internal or kinetic energy of the two water molecules, the total mass stays exactly constant during the reaction. When, however, the energy escapes and the total mass of two H_2O molecules at rest is compared with the total mass of one O_2 and two H_2 molecules, there should be a very small difference (see Fig. 3.4). Because the

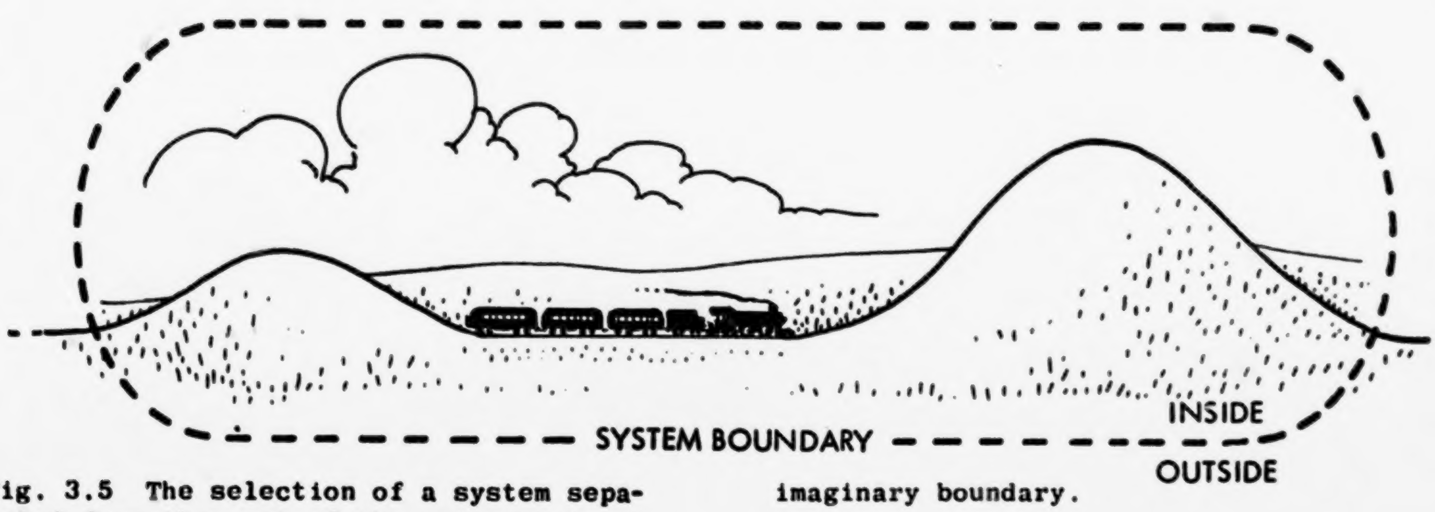


Fig. 3.5 The selection of a system separated from the rest of the universe by an imaginary boundary.

two H_2O molecules have 2×10^{-19} Joules less internal energy, they should have a mass smaller by

$$\Delta m = \frac{\Delta E}{c^2} = \frac{2 \times 10^{-19}}{(3 \times 10^8)^2} \approx 2 \times 10^{-36} \text{ kg}.$$

The fractional difference in the 6×10^{-26} kg total mass of four hydrogen and two oxygen atoms is only about 3×10^{-11}. Even if many water molecules are used to boost the total mass change Δm by a large factor, such a small fractional effect is undetectable with present laboratory techniques. The mass changes in reactions are significant fractions of the total mass only when nuclear potential energies are involved. A slow neutron (almost zero kinetic energy) can be absorbed by the nucleus of U^{235} (an isotope of the element uranium that has, in addition to the 92 protons of any uranium atom, the particular number of 143 neutrons in its nucleus). The resulting nucleus of 236 particles is unstable and splits roughly in half. The result of the reaction is the release of two lighter atomic nuclei, some excess neutrons, and an energy of about 3×10^{-11} Joules. If the energy escapes and the mass of the products at rest is compared with the mass of the reacting particles at rest, the lower internal energy of the products is reflected in a mass decrease of 4×10^{-28} kilograms. In this nuclear reaction the energy release is so large compared with the mass of the particles involved (a total of about 4×10^{-25} kilograms), that the fractional change is significant (0.001 or 0.1%). Of course, the missing mass was not "lost"; it was the mass of the escaped energy. In all reactions where there are energy changes the principle of energy conservation guarantees that mass will be conserved as well, if the mass of each form of energy is properly accounted for. In chemical reactions and most other laboratory experiments the energy changes are sufficiently small compared to the masses involved that the mass of the particles appears to be conserved within experimental error even if the mass-equivalent of the energy involved is ignored.

3.2 ISOLATION, EXTERNAL AND INTERNAL

The principle of energy conservation states that the total energy in the universe remains constant, but in practice the principle is seldom applied to the whole universe. A laboratory experiment or a theoretical calculation usually is confined to one small but interesting part of the universe. It is common to call this piece of the universe a system, and to imagine it completely surrounded by a boundary surface through which pass only things and influences that we know, understand, and can measure (see Fig. 3.5).

The total energy of the universe can be written as the sum of the energy inside and the energy outside the boundary. If no energy crosses the boundary, then we can conclude that the energy of the system (inside) and the energy of the rest of the universe (outside) are separately constant. It is always implicit in such an argument that we know of all forces that might act across the boundary so that we can be really certain that no energy is transferred. When this is the case, the system is called an isolated system.

Even for a system which is not isolated, some useful statement of energy conservation is possible if we can identify and measure all the forms in which energy crosses the boundary. In such circumstances any change ΔE in the energy of the system during a time interval Δt is equal to the net energy that flows inward across the boundary during that time:

$$\begin{aligned}\Delta E &= (\text{energy flow in}) \\ &= (\text{energy flow out}).\end{aligned}$$

With a system like the train of Fig. 3.5, the principle of energy conservation is most useful when it can be written in terms of a few simple quantities. If the train merely coasts up and down hills, the total energy E can be written as

$$E = \tfrac{1}{2}mv^2 + \text{gravitational PE}.$$

Internal energies like those of the parts of the engineer's watch remain separately constant during the coasting, and hence add nothing of interest to the conservation of total energy. If we add to E the energy,

$$E_1 = \text{energy of watch} = \text{constant},$$

the sum $E + E_1$ is a constant but is of no more interest than E itself. It is the flow of energy back and forth from the kinetic energy term ($\frac{1}{2}mv^2$) to the potential energy term that is interesting. Adding another constant term E_1 merely adds complication without adding anything of interest.

If the brakes are applied to the train of Fig. 3.5, the total energy must contain another term E_2 for the heat energy of the brakes. Even in braking situations, the energy

$$E' = \tfrac{1}{2}mv^2 + \text{gravitational PE} + E_2$$

remains constant. If the engine burns fuel in air, this energy is included as well. Calling the chemical energy of the fuel-air mixture E_3, the constant quantity is

$$E'' = \tfrac{1}{2}mv^2 + \text{gravitational PE} + E_2 + E_3.$$

Always the idea is to have as few terms as possible, consistent with all the energy changes that actually occur.

The total energy of any system includes other internal energies in addition to the terms included in a typical description of its "relevant" energy. The watch ticking in the engineer's pocket, the electrons within atoms, the protons within nuclei, etc., all involve energies that are usually neglected in writing down the total energy of a train. These energies are neglected because they do not change separately in the course of the train's motion. In laboratory mechanics experiments it is a great simplification to reduce friction to such a small effect that macroscopic energies are not mixed in with random atomic energies, for then the total energy of the system can be written without including heat energy. Fortunately, the internal energies associated with motions of nucleons within nuclei and with the structures of elementary particles themselves are quite well insulated from the macroscopic world. These energies do not change in macroscopic experiments and need not be included in their description. The only place where the total of all the energy in an object is important, whether it changes or not, is in determining its mass.

3.3 ENERGY-CONSERVATION EXPERIMENTS

The historical development of the law of conservation of energy depended upon the discoveries that various kinds of energy were interchangeable according to unique rules. For a fixed amount of one kind of energy, a fixed amount of another can be obtained. But to make such a rule apparent, one must have a formula for the amount of each kind of energy.

We shall somewhat parallel this historical development in form, though much more efficiently, indeed, if we examine some selected experiments to determine the formulas for different kinds of energy. Briefly, the analysis of each experiment will go something like this:

(1) We assume that there is a law of conservation of energy.[9]

(2) We isolate a system; that is, we attempt to prevent it from exchanging energy with the rest of the universe. Together with assumption (1), this means that we assume the total energy of our isolated system to be a constant.

(3) Furthermore, we choose a system in which our observations lead us to assume that only a few forms of energy are changing, including among them only one kind of energy whose formula we do not know. If E_1, E_2, E_3 represent energies whose formulas we already know, and whose changes we can recognize, and if we assume for our system

$$E_1 + E_2 + E_3 + E_4 = \text{constant}, \quad (3.13)$$

then we can calculate changes in E_4:

$$\Delta E_4 = -(\Delta E_1 + \Delta E_2 + \Delta E_3). \quad (3.14)$$

(4) If we can obtain from Eq. (3.14) a formula for E_4 (like $KE = \frac{1}{2}mv^2$) in terms of all the variables that describe the system (length, temperature, etc.), and if the same formula describes all such experiments, then we have confirmed the conservation law, Eq. (3.13), limited to these four kinds of energy.

The confirmation of Eq. (3.13) supports indirectly our faith in the assumptions (1), (2), and (3) used to obtain it. Of course, each such experiment also helps confirm that we have the correct formulas for E_1, E_2, and E_3.

The fact that we can for so many different cases find a conservation law of the form of Eq. (3.13), summing many and various kinds of energy to a constant total, leads us to believe that there is a quantity corresponding to what we call energy. And it does seem to be conserved. Whenever we keep careful account, none is found to be created or destroyed.

We shall see some examples of this kind of analysis in the following sections, beginning with kinetic energy.

3.4 KINETIC ENERGY

In this section we shall start with the assumption that a mass m moving with speed v has a kinetic energy determined by these two numbers. By considering cases in which kinetic energy seems to be conserved, we shall show that if it is conserved in any collisions (called elastic collisions), it must have the form

$$KE = (\text{constant})\, mv^2 .$$

The development will reveal connections among kinetic energy, momentum conservation, and Galilean relativity. Let two observers, moving with constant relative velocity, watch the same collision. To show that they agree on whether or not energy is conserved in the collision, we shall need the law of conservation of momentum.

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[9] Historically, the grand conservation law encompassing all the energy forms did not come until the late nineteenth century. But it was already known around 1700 that there are some simple systems in which the kinetic energy is conserved. Early studies of heat energy (then interpreted as "caloric") separate from mechanical energy invoked a separate conservation law. We now know that the two laws are special cases of the energy conservation law.

Eventually Eq. (3.1) must meet additional tests. In experiments in which KE from Eq. (3.1) is conserved, does the rest of the energy also remain constant, as predicted by the law of conservation of energy? In additional experiments involving other forms of energy as well, does this formula permit writing an energy conservation law that is confirmed by measurements?

A mass m_A, moving with velocity $\vec{v}_{Ai}$, strikes a body of mass m_B, which had initially a velocity $\vec{v}_{Bi}$. This is the start of a collision. We shall show that there are some such collisions in which kinetic energy is "obviously" present in equal amounts before and after the collision, even to one who does not know Eq. (3.1). There are indeed such collisions. They occur very frequently in the microscopic realms of atomic and nuclear physics. A macroscopic collision in which kinetic energy is "obviously" the same afterwards as before can be demonstrated in the class room on the air track. Collisions which do not change the kinetic energy are called elastic collisions.

3.4.1 The Equal-Mass Elastic Collision.

Two identical gliders A and B are placed on the air track, B initially at rest and A with initial velocity $\vec{v}_{Ai}$. The gliders have equal mass,

$$m_A = m_B = m,$$

and are equipped with spring-steel bumpers on the ends. These bumpers can be distorted in a collision and returned to their original shapes with very little transfer of energy to the motion of the atoms within the springs. The measurements will show that this is so.

	TRIAL NO.	1	2	3
BEFORE	$\vec{v}_{Ai}$	0.115	0.573	1.25
AFTER	$\vec{v}_{Af}$	≈zero	≈zero	≈0.01
	$\vec{v}_{Bf}$	0.114	0.565	1.24

Units are m/sec

Table 3.1 Elastic collisions for data for

$$m_A = m_B = 0.400 \text{ kg}$$
$$v_{Bi} = 0$$

initial and final velocities are presented for three trials.

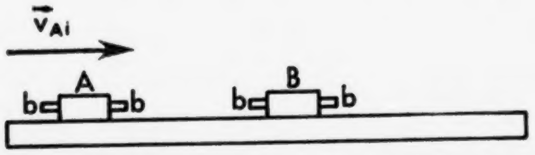
Some of the data from a typical experiment might look like that in Table 3.1. The results of such an experiment would be well described by the equations

$$\vec{v}_{Af} = 0, \qquad (3.15a)$$

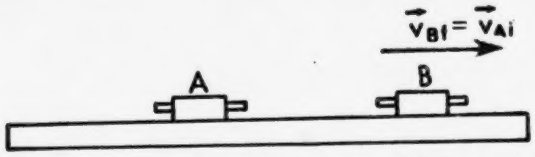
$$\vec{v}_{Bf} = \vec{v}_{Ai}. \qquad (3.15b)$$

The before and after pictures of such an elastic collision between equal masses would resemble Fig. 3.6.

With the assumption that the kinetic energy of a moving mass depends upon m and v only, Eqs. (3.15) tell us that kinetic energy is con-



(a) Two identical gliders A and B before the collision. Glider B is at rest while glider A moves with velocity $\vec{v}_{Ai}$. The gliders are equipped with elastic steel bumpers b. The kinetic energy is contained in the motion of mass m_A at speed $|\vec{v}_{Ai}|$.



(b) After the collision A is found at rest and B moves with velocity $\vec{v}_{Bf} = \vec{v}_{Ai}$. Because the two gliders are identical, it is here "obvious" that B carries off the same kinetic energy originally brought to the collision by A. A collision in which kinetic energy is conserved is called an elastic collision.

Fig. 3.6 An elastic collision between equal masses.

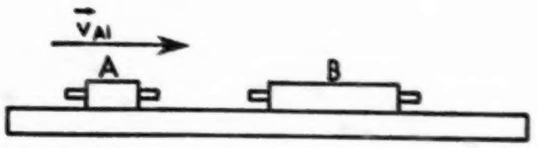


Fig. 3.7a The unequal masses A and B on the air track before the collision. Initially

$$\vec{v}_A = \vec{v}_{Ai}$$
$$\vec{v}_B = 0.$$

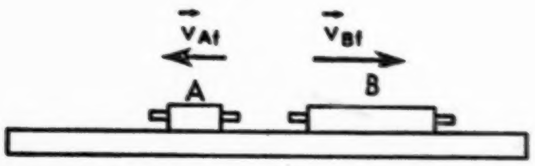


Fig. 3.7b The unequal masses A and B separating after the collision. Finally

$$\vec{v}_A = \vec{v}_{Af}$$
$$\vec{v}_B = \vec{v}_{Bf}.$$

served in this experiment. Initially a mass $m_A = m$ travels with speed $|\vec{v}_{Ai}|$, and an equal mass sits at rest. Finally, the data describe a situation in which mass $m_B = m$ travels with speed $|\vec{v}_{Ai}|$, and an equal mass sits at rest. The kinetic energy is unchanged in this collision.

This experiment serves to demonstrate that we can find a set of "elastic" springs. But an experiment with equal masses will not demonstrate that the kinetic energy must have the unique form of Eq. (3.1). Just for fun, try calculating from the data the initial and final kinetic energies on the following assumptions:

$$\text{Try: } KE' \propto (m_A\,|\vec{v}_A| + m_B\,|\vec{v}_B|) \qquad (3.16a)$$

$$\text{Try: } KE' \propto (m_A\,|\vec{v}_A|^2 + m_B\,|\vec{v}_B|^2) \qquad (3.16b)$$

$$\text{Try: } KE' \propto (m_A\,|\vec{v}_A|^3 + m_B\,|\vec{v}_B|^3) \qquad (3.16c)$$

They all are conserved. Can you see why this particular experiment will not distinguish among them?

Not all collisions are elastic, but one very simple collision experiment, in which kinetic energy is "obviously" conserved, has been described here. Equivalent experiments can be performed in other ways. Two air gliders with mutually repelling magnets will collide elastically, as did the gliders with spring-steel bumpers. If they have equal masses and one in motion is sent against another at rest, they will exchange velocities in the magnetic collision. The one initially in motion will stop, and the other acquire the initial velocity.

A proton and a neutron have nearly equal masses. Quite often when a neutron in motion collides "head-on" with a proton at rest ("head-on" assures straight-line motion), it comes to rest, and the proton goes on ahead with the neutron's total kinetic energy; i.e., with the same mass and the same speed.

3.4.2 The Unequal-Mass Elastic Collision.

The same elastic springs employed in our first experiment can be used in another experiment with unequal masses. We shall find the conservation of energy in the slightly more complicated collision just as convincing (after some analysis) from the data we record. As a reward for carrying through the more complex analysis, we shall be able to determine the actual form, Eq. (3.1), for the kinetic energy.

This second experiment begins with mass m_B at rest on the air track.[10] The initial velocity of mass m_A is $\vec{v}_{Ai}$. The two gliders have different masses, but they use the same spring bumpers used so successfully before.

A few of the data from such an experiment, pictured in Fig. 3.7, might look like those in Table 3.2.

A quick inspection of the data in Table 3.2 is not sufficient to see

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[10] We choose m_B initially at rest for simplicity. It will turn out that there is nothing to be gained but complication by choosing a nonzero v_{Bi}, unless we could set

$$\vec{v}_{Bi} = -\frac{m_A}{m_B}\vec{v}_{Ai},$$

to make the laboratory the zero-momentum (center-of-mass) frame.

whether kinetic energy is the same before and after the collision.[11] But look at the experiment from the center-of-mass viewpoint! Table 3.3 shows the same glider motions from the center-of-mass frame. Look at the constancy of kinetic energy! In the center-of-mass frame, each mass leaves the collision with the same speed that it entered; only its direction is changed.

If the kinetic energy depends only on the mass and the speed, it must surely be unchanged before and after these collisions described in Table 3.3. The data of this table can be described as far as energy is concerned by

$$|v_{Bf}'| = |v_{Bi}'| \qquad (3.17a)$$

$$|v_{Af}'| = |v_{Ai}'|. \qquad (3.17b)$$

For collisions which obey Eqs. (3.17), it is "obvious" to an observer in the center-of-mass frame that kinetic energy is the same after the collision as before.

The question we now want to answer is "What quantity remains constant in the laboratory frame before and after collisions which are 'obviously' elastic in the center-of-mass frame?"

We cannot try all possible functions of m and v in the time allotted. But we might compare the three assumptions of Eqs. (3.16):

(a) Is the quantity $(m_A|v_A| + m_B|v_B|)$ constant? (Table 3.4a)

(b) Is the quantity $(m_A|v_A|^2 + m_B|v_B|^2)$ constant? (Table 3.4b)

(c) Is the quantity $(m_A|v_A|^3 + m_B|v_B|^3)$ constant? (Table 3.4c)

	TRIAL NO.	1	2	3	4
BEFORE	$\vec{v}_{Ai}$	0.247	0.553	1.07	1.54
AFTER	$\vec{v}_{Af}$	-0.104	-0.235	-0.450	-0.651
	$\vec{v}_{Bf}$	0.140	0.314	0.601	0.870

Units are m/sec

Table 3.2 Velocity data from an air-track collision. The collisions between unequal masses use spring steel bumpers that are supposed to make them elastic. The data will tell if this is so. The initial conditions are

$$m_A = 0.200 \text{ kg}, \ \vec{v}_A = \vec{v}_{Ai}$$

$$m_B = 0.500 \text{ kg}, \ \vec{v}_B = 0.$$

	TRIAL NO.	1	2	3	4
BEFORE	$\vec{v}_{Ai}'$	0.176	0.395	0.765	1.10
	$\vec{v}_{Bi}'$	-0.0706	-0.158	-0.306	-0.440
AFTER	$\vec{v}_{Af}'$	-0.175	-0.393	-0.756	-1.09
	$\vec{v}_{Bf}'$	0.0694	0.156	0.295	0.430

Units are m/sec

Table 3.3 The velocities of Table 3.2 from the center-of-mass frame. The center of mass moves in the laboratory with velocity

$$\vec{u} = \frac{(m_A)}{(m_A + m_B)}\vec{v}_{Ai} = \frac{2}{7}\vec{v}_{Ai}.$$

Table 3.4 shows that in these collisions for which kinetic energy is conserved in the center-of-mass system the quantity $(m_A|\vec{v}_A|^2 + m_B|\vec{v}_B|^2)$ is conserved in the laboratory system. This is the first justification for the form of Eq. (3.1)

$$KE = \tfrac{1}{2}m_A|\vec{v}_A|^2 + \tfrac{1}{2}m_B|\vec{v}_B|^2$$

The factor $\frac{1}{2}$ arises from the relation between force and changes of kinetic energy. Any other multiple of mv^2 would also be conserved in an elastic collision.

3.4.3 The Form of the Kinetic Energy.

It is possible to show mathematically that a collision for which

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[11] Actually a very clever observer might notice a clue to the elasticity in Table 3.2 in that the size of the relative velocity between m_A and m_B is the same after the collision as before. But the center-of-mass frame gives a better vantage point for viewing the elastic collision.

	TRIAL NO.	1	2	3	4
BEFORE	$m_A\|v_{Ai}\|$	0.0494	0.111	0.214	0.308
AFTER	$m_A\|v_{Af}\| + m_B\|v_{Bf}\|$	0.0908	0.204	0.390	0.595

Table 3.4a Is the quantity $(m_A|\vec{v}_A| + m_B|\vec{v}_B|)$ the same before and after the collision?

	TRIAL NO.	1	2	3	4
BEFORE	$m_A\|v_{Ai}\|^2$	0.0122	0.0612	0.230	0.474
AFTER	$m_A\|v_{Af}\|^2 + m_B\|v_{Bf}\|^2$	0.0120	0.0605	0.222	0.463

Table 3.4b Is the quantity $(m_A|v_A|^2 + m_B|v_B|^2)$ the same before and after the collision?

	TRIAL NO.	1	2	3	4
BEFORE	$m_A\|v_{Ai}\|^3$	0.00302	0.0338	0.246	0.684
AFTER	$m_A\|v_{Af}\|^3 + m_B\|v_{Bf}\|^3$	0.00160	0.0181	0.127	0.384

Table 3.4c Is the quantity $(m_A|v_A|^3 + m_B|v_B|^3)$ the same before and after the collision?

$$\vec{v}'_{Bf} = -\vec{v}_{Bi} \tag{3.18a}$$

$$\vec{v}'_{Af} = -\vec{v}_{Ai} \tag{3.18b}$$

in the center-of-mass frame will not change the quantity

$$m_A|\vec{v}_A|^2 + m_B|\vec{v}_B|^2$$

in any other inertial frame.

The center-of-mass frame is an inertial frame in which the center-of-mass point moves with the constant velocity zero.

The center of mass moves with constant velocity $\vec{U}$ with respect to another.inertial frame. In this frame, the initial and final velocities for the elastic collision described by Eqs. (3.18) are

$$\vec{v}_{Ai} = \vec{v}'_{Ai} + \vec{U};\ \vec{v}_{Af} = \vec{v}'_{Af} + \vec{U} \tag{3.19a}$$

$$\vec{v}_{Bi} = \vec{v}'_{Bi} + \vec{U};\ \vec{v}_{Bf} = \vec{v}'_{Bf} + \vec{U}. \tag{3.19b}$$

In order to write the quantity

$$m_A|\vec{v}_{Ai}|^2 = m_A|v'_{Ai} + U|^2$$

in the laboratory frame, we need to do a little geometry. Figure 3.8 shows that the length $|\vec{v}' + \vec{U}|$ of the vector, $\vec{v} + \vec{U}$, is either the sum of the lengths if the vectors are parallel, or the difference of the lengths if they are opposite.

$$|\vec{v}' + \vec{U}| = \begin{cases} |\vec{v}'| + |\vec{U}| \text{ if } \vec{v}' \text{ and } \vec{U} \text{ are parallel} \\ |\vec{v}'| - |\vec{U}| \text{ if } \vec{v}' \text{ and } \vec{U} \text{ are opposite and if } |v'| > |\vec{U}| \\ |\vec{U}| - |\vec{v}'| \text{ if } v' \text{ and } U \text{ are opposite and if } |\vec{U}| > |\vec{v}'|. \end{cases} \tag{3.20}$$

Hence

$$|v' + U|^2 = \begin{cases} |\vec{v}'|^2 + |2\vec{v}'|\ |\vec{U}| + |U|^2 \\ \text{if } \vec{v} \text{ and } \vec{U} \text{ are parallel} \\ |\vec{v}'|^2 - 2|\vec{v}'|\ |\vec{U}| + |\vec{U}|^2 \\ \text{if } \vec{v}' \text{ and } \vec{U} \text{ are opposite.} \end{cases} \tag{3.21}$$

To allow for these two cases with one simple notation, we introduce the "scalar product" or "dot product" of two one-dimensional vectors:[12]

$$\vec{v}' \cdot \vec{U} = \vec{U} \cdot \vec{v}' = \begin{cases} + |\vec{v}'|\ |\vec{U}| \text{ for } \vec{v}' \text{ parallel to } \vec{U} \\ - |\vec{v}'|\ |\vec{U}| \text{ for } \vec{v}' \text{ opposite to } \vec{U}. \end{cases} \tag{3.22}$$

This notation permits us to write

$$|\vec{v}'|^2 = \vec{v}' \cdot \vec{v}' \tag{3.23}$$

and

$$|\vec{v}' + \vec{U}|^2 = |\vec{v}'|^2 + 2\vec{v}' \cdot \vec{U} + |\vec{U}|^2. \tag{3.24}$$

Notice how much simpler it is to write Eq. (3.24) instead of Eq. (3.21). The scalar product has a pair of useful properties which can be verified from the definition, Eq. (3.22): multiplication by a scalar,

$$m\vec{v}' \cdot \vec{U} = (m\vec{v}') \cdot \vec{U} = \vec{v}' \cdot (m\vec{U}). \tag{2.35}$$

and addition

$$(\vec{v}'_A + \vec{v}'_B) \cdot \vec{U} = \vec{v}'_A \cdot \vec{U} + \vec{v}'_B \cdot \vec{U}. \tag{3.26}$$

Before the collision

$$\begin{aligned} m_A|\vec{v}_{Ai}|^2 + m_B|\vec{v}_{Bi}|^2 &= m_A|\vec{v}'_{Ai} + \vec{U}|^2 \\ &\quad + m_B|\vec{v}'_{Bi} + \vec{U}|^2 \\ &= m_A|\vec{v}'_{Ai}|^2 + 2m_A\vec{v}'_{Ai} \cdot \vec{U} \\ &\quad + m_A|\vec{U}|^2 + m_B|\vec{v}'_{Bi}|^2 \\ &\quad + 2m_B\vec{v}'_{Bi} \cdot \vec{U} + m_B|\vec{U}|^2. \end{aligned} \tag{3.27}$$

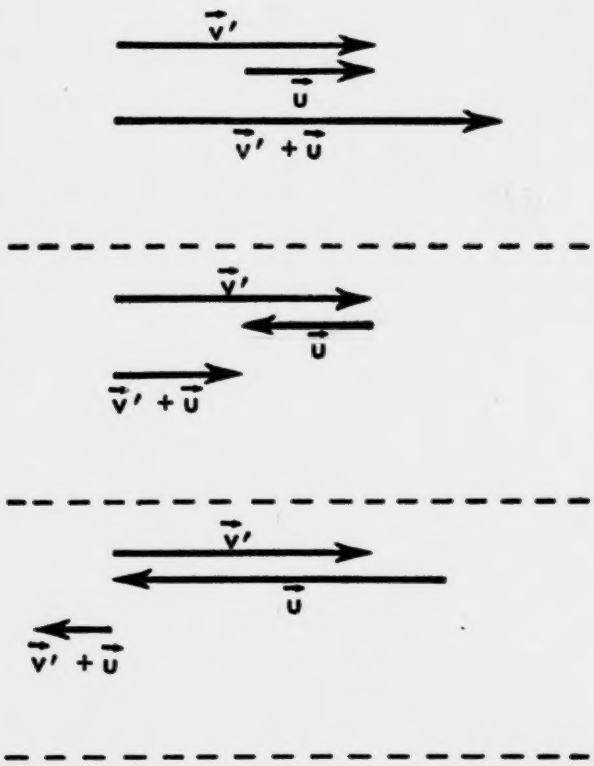

Fig. 3.8 The formula for the length of the vector $(\vec{v}' + \vec{U})$ depends on whether $\vec{v}'$ and $\vec{U}$ are parallel or antiparallel

$$\begin{aligned} \text{I: } |\vec{v}' + \vec{U}| &= |\vec{v}'| + |\vec{U}| \\ \text{II: } |\vec{v}' + \vec{U}| &= |\vec{v}'| - |\vec{U}| \\ \text{III: } |\vec{v}' + \vec{U}| &= |\vec{U}| - |\vec{v}'|. \end{aligned}$$

Collecting appropriate terms together, we find that

$$\begin{aligned} m_A|\vec{v}_{Ai}|^2 + m_B|\vec{v}_{Bi}|^2 &= m_A|\vec{v}'_{Ai}|^2 \\ &\quad + m_B|\vec{v}'_{Bi}|^2 + 2(m_A\vec{v}'_{Ai} \\ &\quad + m_B\vec{v}'_{Bi}) \cdot \vec{U} + (m_A + m_B)|\vec{U}|^2. \end{aligned} \tag{3.28}$$

After the collision, the same expansion and collection of terms gives

$$\begin{aligned} m_A|\vec{v}_{Af}|^2 + m_B|\vec{v}_{Bf}|^2 &= m_A|\vec{v}'_{Af}|^2 \\ &\quad + m_B|\vec{v}'_{Bf}|^2 + 2(m_A\vec{v}'_{Af} \\ &\quad + m_B\vec{v}'_{Bf}) \cdot \vec{U} + (m_A + m_B)\ |\vec{U}|^2. \end{aligned} \tag{3.29}$$

For elastic collisions defined by Eq. (3.18), the two expressions in Eqs. (3.28) and (3.29) are equal. Because of Eq. (3.18), each of the first two terms on the right side of Eq. (3.28) is separately equal to each of the first two terms on the right side of Eq. (3.29):

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[12] The same notation will be used later for the three-dimensional generalization, where the term "scalar product" is especially appropriate. For one-dimensional vectors expressed by positive and negative numbers it is just like the ordinary arithmetical product.

$$m_A|\vec{v}'_{Ai}|^2 = m_A|\vec{v}'_{Af}|^2, \qquad (3.30a)$$

$$m_B|\vec{v}'_{Bi}|^2 = m_B|\vec{v}'_{Bf}|^2. \qquad (3.30b)$$

Because momentum is conserved,

$$m_A\vec{v}'_{Ai} + m_B\vec{v}'_{Bi} = m_A\vec{v}'_{Af} + m_B\vec{v}'_{Bf}. \qquad (3.31)$$

Hence, the third term on the right side of Eq. (3.28) is equal to the third term on the right side of Eq. (3.29):

$$2(m_A\vec{v}'_{Ai} + m_B\vec{v}'_{Bi}) \cdot \vec{U} = 2(m_A\vec{v}'_{Af} + m_B\vec{v}'_{Bf}) \cdot \vec{U}. \qquad (3.32)$$

The last term on the right-hand side of Eq. (3.28) is identical with the last term on the right hand side of Eq. (3.29).

Let us summarize what we have just shown. If there are in fact elastic collisions described in the center-of-mass frame by merely a reversal of velocities Eq. (3.18), and if momentum is conserved Eq. (3.31), then the quantity $m_A|\vec{v}_A|^2 + m_B|\vec{v}_B|^2$ in any other inertial frame is the same after the collision as it was before it.

Hence, if there is a kinetic energy associated with matter in motion, and if we can write a conservation equation for kinetic energy alone in elastic collisions, the formula for kinetic energy must be

$$KE = (\text{constant})\ mv^2.$$

3.4.4 The Ultimate Appeal.

Our few experiments and their analyses only indicate but do not prove that kinetic energy is proportional to mv^2. Nor is it at all justified to jump to any grand conclusion yet that there is such a thing as energy which is conserved.

But there are a few things that we can do to check our hypothesis that kinetic energy is the same after an elastic collision as before. We can check the other properties (besides velocity) of the isolated air-track gliders before and after the experiment to see if they are the same. If other properties (temperature, shape, etc.) of the gliders stay the same in such collisions, perhaps that indicates that the other energies, internal to the glider, remain constant

The next step in the development of energy conservation is to go back to the laboratory, to put the equation

$$KE = \tfrac{1}{2}mv^2$$

to an experimental test. This can be done by studying experiments in which kinetic energy is exchanged for another form. The laboratory provides the ultimate test for physical theories.